REPORT RESUMES

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ELEMENTARY ARITHMETIC AND LEARNING AIDS.
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THIS MANUAL WAS DESIGNED TO HELP TEACHERS TO ACQUIRE THE KNOWLEDGE AND SKILLS NECESSARY FOR THE EFFECTIVE TEACHING OF MODERN MATHEMATICS. THE SPECIFIC PURPOSE OF THIS MANUAL IS TO IDENTIFY SOME OF THE CONTENT WHICH IS FUNDAMENTAL TO THE USE OF LEARNING AIDS AND TO SUGGEST SPECIFIC EXAMPLES FOR THE USE OF VISUAL AND MANIPULATIVE DEVICES IN ELEMENTARY CLASSROOM PRESENTATIONS. MATERIALS FOR AN INSERVICE COURSE IN MATHEMATICS ARE DEVELOPED FOR (1) ARITHMETIC AS A SYMBOLIC LANGUAGE, (2) OUR OWN NUMERATION SYSTEM, (3) OTHER SYSTEMS AND BASES, (4) TEACHING THE OPERATIONS WITH LEARNING AIDS, (5) FRACTIONS, AND (6) INFORMAL GEOMETRY AND MEASUREMENT. THE PUBLICATION IS INTENDED FOR STATE SUPERVISORS OF MATHEMATICS AND CLASSROOM TEACHERS. THIS DOCUMENT IS AVAILABLE FOR $1.00 FROM THE SUPERINTENDENT OF DOCUMENTS, U.S. GOVERNMENT PRINTING OFFICE, WASHINGTON, D.C. 20402. (RP)

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and LEARNING AIDS

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OE-29063
Circular No. 784

ELEMENTARY ARITHMETIC

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and LEARNING AIDS

By

Patricia M. Spross, Specialist
Division of Plans and Supplementary Centers

U.S. DEPARTMENT OF
HEALTH, EDUCATION, AND WELFARE
JOHN W. GARDNER, *Secretary*

Office of Education
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FOREWORD

The history of mathematics, the development and structure of other numeration systems, and computation in other bases are being introduced with increasing success in the elementary grades. Such concepts help children to gain insight into mathematical principles; they give meaning to heretofore rote responses, encourage the gifted to greater academic achievement, and stimulate children's interest and enjoyment of mathematics. Elementary students can thus master not only the "facts" and computational skills but gain an increasing understanding of them as well.

Since today's crowded curriculum must be compressed into the instructional time which is available, it is necessary to find a way to teach more efficiently. New learning aids are being developed to assist teachers of elementary arithmetic in making classroom presentations that are meaningful and mathematically sound; yet teachers must have sufficient mathematical background to use such aids in an effective manner. The purpose of this manual is to point out some of the content which is fundamental to the use of learning aids and to suggest specific examples for the use of visual and manipulative devices in elementary classroom presentations.

Teachers who have little background in mathematics need to be encouraged to strengthen their competency in content and to gain confidence in the methodology which accompanies curriculum change. The materials of Elementary Arithmetic and Learning Aids are designed to help teachers in their inservice programs to acquire the knowledge and skills necessary for the effective teaching of modern mathematics.

Both the instructional equipment and materials and the State supervisory services available under title III of the National Defense Education Act are helping to strengthen the school programs of mathematics. This publication is a service of the Instructional Resources Branch, U.S. Office of Education, to the State supervisors of mathematics and to the classroom teachers who are guiding our future citizens toward mathematical literacy.

The teaching techniques and student activities included in this publication were first developed by the author for the Lansing, Mich., elementary inservice programs in the use of learning aids purchased under title III, National Defense Education Act. Acknowledgement is given to Forrest G. Averill, Lansing Superintendent of Schools, for permission to use the original manuscript as a basis for this manual. Appreciation is also expressed to Harry L. Phillips, Specialist in Mathematics, Division of State Grants, U.S. Office of Education, who gave valuable assistance in the preparation of the manual; and to Seaton E. Smith, Jr., Mathematics Specialist, State Department of Education, W.Va., for his many helpful suggestions.

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PART I. ARITHMETIC AS A SYMBOLIC LANGUAGE

USING SYMBOLS TO REPRESENT QUANTITY

Numeration is an organized system of expressing quantity with symbols. The symbols are called numerals. The *idea* of quantity is the number. Obviously there are times when we wish to talk and write about numbers. Whenever it seems appropriate we shall use the term number. Many types of numerals have been used throughout the history of mathematics. It has generally been true that man has always tried to make whatever numeration system he used simpler and more convenient. In this sense, we might say that arithmetic just "grew." It has grown for some five or six thousand years. Many older numeration systems might not be recognized as such today.

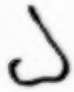
Counting is basic to arithmetic. It might be possible, if time permitted, to count to the answer in an arithmetic problem. Since mathematics should be simple and convenient in order to be of the most use, we count in groups rather than by naming every separate quantity. We count in groups of tens and call this a *base ten* numeration system. There exist records of some very simple numeration systems like this:

(/)	one
(//)	two
(///)	some
(any amount of more than /// with no symbol to represent it)	many

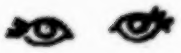
It makes little difference what the symbols are or how they are named so long as the quantity they represent is agreed upon. We can see that in this system four could be either *many*, or *some plus one*, or *one plus some*, or perhaps just *onesome* as a word. This would be like our number word twenty-one. When we say "twenty-one" we think of an amount, and seldom stop to think that it is equal to 2 tens plus 1 one.

In the history of numeration, quantities have been represented in many ways. The Romans and Greeks used letters and the Egyptians used pictures. Often the symbol has had meaning or some relationship to the quantity which it represented. For instance, a nose could represent one.

A numeration system might develop in this manner:

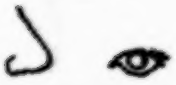


Nose represents one. (Everyone has just one of these.)

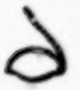


Eyes come in pairs. (Eyes have often represented two. Sometimes wings have been used.)

Drawing two eyes every time is a bother, so one eye might come to represent two all by itself.



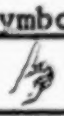
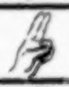
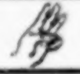
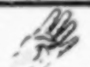
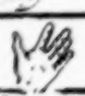
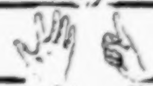
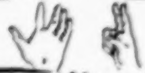
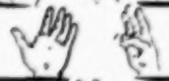
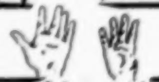
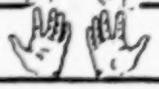
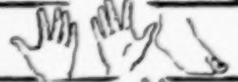
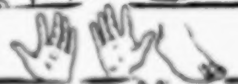
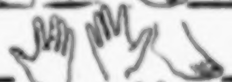
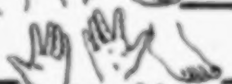
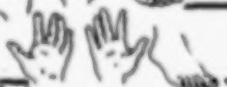
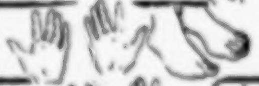
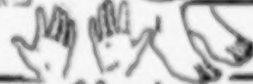
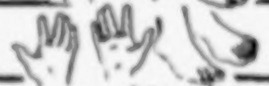
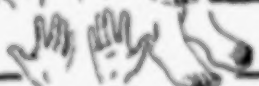
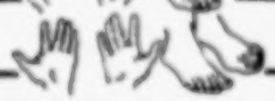
Nose-eye could represent three.



It is rather a nuisance to have to make these two separately. The symbol for three might come to be no more than something like this.

A counting system has been devised by naming fingers and toes and then combining these names. In this way counting is done in groups of five instead of ten as we do. This can be done until twenty is reached and then groups of twenty can be used.

A finger and toes counting system might look like this:

amount	symbol	number name
1		1 finger
2		2 fingers
3		3 fingers
4		hand less 1 finger
5		hand
6		hand and 1 finger
7		hand and 2 fingers
8		hand and 3 fingers
9		2 hands less 1 finger
10		2 hands
11		2 hands and 1 toe
12		2 hands and 2 toes
13		2 hands and 3 toes
14		2 hands and 4 toes
15		2 hands and 1 foot
16		2 hands and 1 foot and 1 toe
17		2 hands and 1 foot and 2 toes
18		2 hands and 1 foot and 3 toes
19		2 hands and 1 foot and 4 toes
20		2 hands and 2 feet

ACTIVITIES WITHOUT LEARNING AIDS

Such a system of naming numbers is useful in helping children to understand the representation of quantity by the use of a symbol. A lesson-game for primary children could consist of the following:

Ask children: "How many things would these words represent?"

hand + hand + 1 toe?

hand + hand + 4 toes?

hand + hand + foot?

hand + hand + foot + 1 toe?

hand + hand + foot + foot?

Let children do exercises by thinking of a hand as 5, each foot as 5, and each digit as 1. It will help to have youngsters "make a fist" and sit with hands on their desks or in their laps to encourage them to think in groups of five. Give students ample time to think during each exercise. In order to avoid misrepresenting more complicated operations of addition and subtraction, ask if each of these pairs of expressions would represent the same quantity. Give youngsters an opportunity to explain how they think of the grouping in each exercise, as indicated by the parentheses.

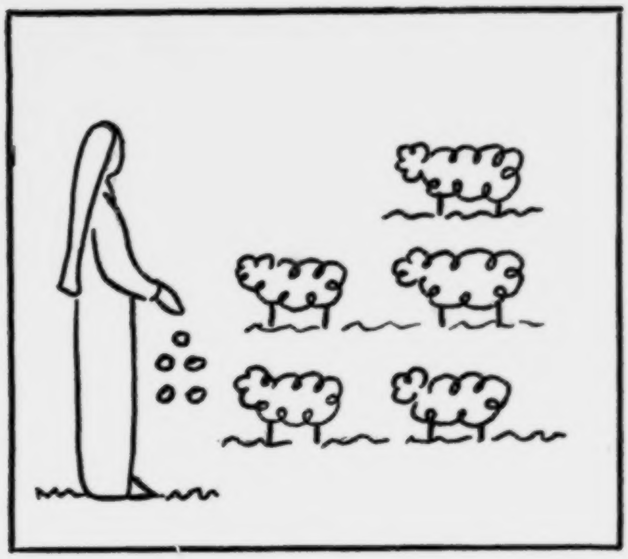
(hand + hand) - finger = ?	hand - (3 fingers - 2 fingers) = ?
(5 + 5) - 1 = ?	5 - (3 - 2) = ?
10 - 1 = 9	5 - 1 = 4
hand + (hand - finger) = ?	(hand - 3 fingers) - 2 fingers = ?
5 + (5 - 1) = ?	(5 - 3) - 2 = ?
5 + 4 = 9	2 - 2 = 0

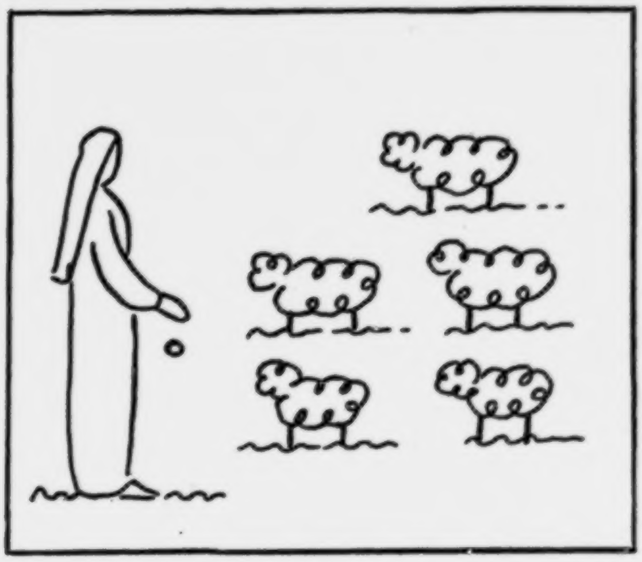
Great care should be taken to avoid mathematical fallacies when playing this or other games in arithmetic. On the other hand, it is possible to develop sound principles when games are used carefully. Small children may use this very simple numeration system to aid them in computing mental arithmetic problems. Develop additional exercises suitable to the children's abilities. Always encourage youngsters to contribute their own exercises and to discuss these with the class.

In order to demonstrate the development and use of abstract symbols, let the children imagine that they need to "keep track of something" but that they know of no system of numeration to use. Ask them to make up symbols to represent amounts and then to do a problem or two in their own system. Suggest simple problems such as 4 +3 =7 or 8 - 5 =3. Have students do each problem in both our own system and in the invented one. Let students develop their own algorisms -- the way they put it on paper -- in order to perform the computation. Discuss which may be the best way. From this demonstration students will discover that a numeration system should be simple, convenient, and consistent. They should realize that the assignment of a numeral to a quantity is arbitrary. Any symbol may be used but its use must be consistent once it is established. Students can be convinced of this by letting them exchange numeration systems with their neighbors, and attempt to learn to compute with the use of unfamiliar numerals. This activity may be done at any grade level.

LEARNING AIDS FOR COUNTING

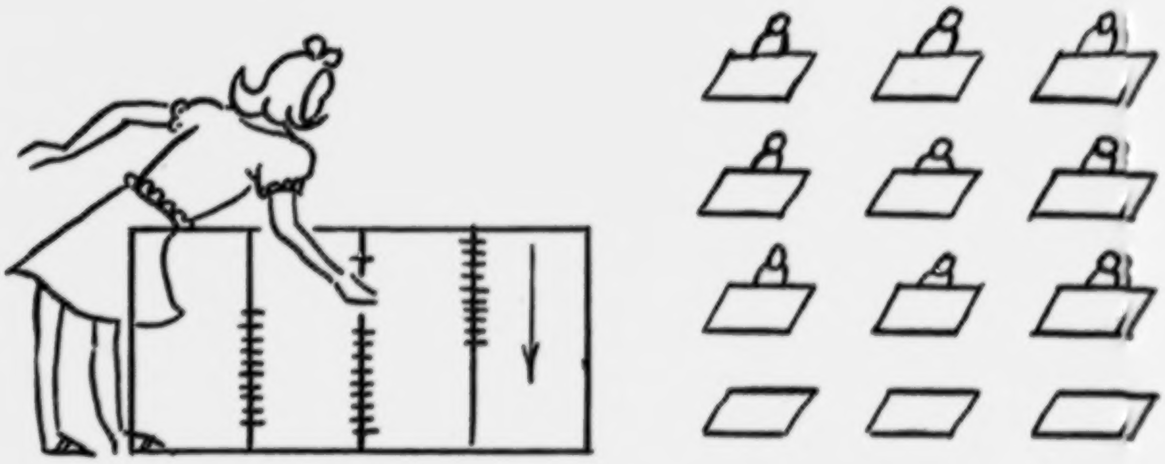
Make a panorama to show how a shepherd might keep track of sheep by using 1 pebble to represent each sheep. Use animal crackers mounted on small squares for the sheep. Flannelboard cutouts could also be used. Show 1-1 and 1-5 relationships.





From examples such as those above, show that there might be a need to change the quantity ascribed to each pebble if there were a change in the number of animals. For example, it would be difficult to represent 7 sheep with pebbles having a value of 5 each. Let students experiment and devise methods as the number to be represented is changed.

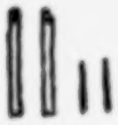
Have each student bring a pebble to school. Place these in a box near the classroom door. Take attendance by matching one student with one pebble. Point out that such mapping of students to pebbles does not require that we have a name for the amount. The pile of pebbles represents the number of students in the room. If there are any remaining, someone is absent; if there are not enough, someone is there who should not be (or someone took more than one pebble). The same activity can be carried out with a counting frame, sticks, or with an abacus. When the tenth student comes into the room, he slides down the 10 one beads and exchanges them for 1 ten bead; the next child shows eleven on the abacus (a 1 ten bead on the tens' column and 1 one bead on the ones' column).



Compare activities of this nature with our need to have counting names so that one can state the number of students present. Point out the advantages or disadvantages of both methods. For example, the number name "twelve" conveys more meaning to us than a pile containing this amount of pebbles or sticks but having no number name to describe it.

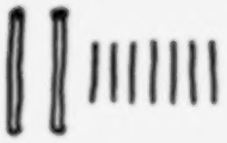
Counting sticks and other counting devices may be used to develop the understanding of a need for naming quantities. Demonstrate that in a 1-1 relationship, 1 pebble represents 1. Show that this would be inconvenient if a large number were to be accounted for. Show the advantage of a 1-5 relationship by counting 5 sheep for each pebble as above. Do this for a 1-10 relationship. Create a more difficult situation by asking students how 62 sheep could be represented. Let youngsters experiment in determining the amount each stick or pebble would need to represent. Suggest that if a 1-20 relationship is used some sticks would have to represent 20 and some other sticks would have to represent one. Use sticks of different colors or lengths. Select quantities that are suited to students' abilities.

Set up displays of counting sticks and let the children tell how many things are represented by the display.

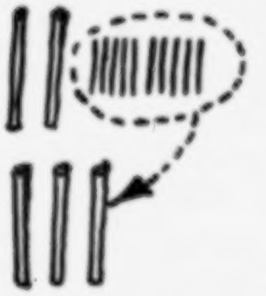


Each long stick represents 20 sheep, each short stick one. How many sheep are represented? (20 + 20 + 1 + 1 = 42)

After children are convinced that our numeration system is an arbitrary assignment of a given quantity to a mutually agreed upon symbol, relate the above activities to our base ten system. Make a display with counting sticks that represents a base ten numeral such as 27.



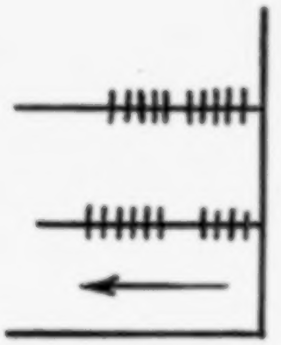
Each long stick represents ten, each short stick represents one. How much is represented? (10 + 10 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 27)



How many more short sticks would need to be added before we could use three long sticks to represent an amount. What would be the amount? (Three more short sticks would need to be added.)

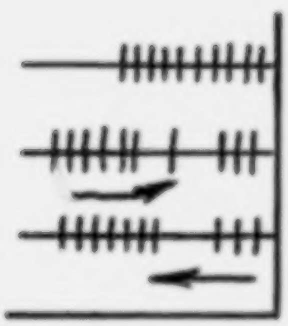
Show that 10 + 10 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 30 or three groups of ten.

Similar activities can also be carried out on a flannelboard using cutouts. Let students demonstrate the representation of various quantities, using sticks or counters to which a value has been assigned. Numerical understandings as presented in primary books may be developed in this manner. One student may represent the quantity with counting sticks, discs, or any suitable device; another student may write the amount on the chalkboard using the correct numeral, or may select the numeral from a display.



Similar activities can be done with a counting frame. Ask a child to represent 6 on the counting frame. After this has been done ask such questions as: "How do we know this is 6?" (We can count the beads.) "What can we find out about 6?" (It is made up of 3 ones plus 3 ones, or 3 + 3; it is 5 + 1; it is 6 ones, etc.)

Encourage students to manipulate the beads themselves in order to make other discoveries. Let them symbolize each statement on the chalkboard.



Let a child represent 7 on the frame. Ask: "What can we do to 7 in order to get to 6?" Encourage students to symbolize 7 - 1 = 6 after it has been discussed. Do not penalize incorrect answers. They may be used for a revealing and worthwhile class discussion. Let the class discover the reason for errors. Use the counting frame to let students demonstrate the basis for their thinking in correcting their own errors.

Learning aids that may be collected, counted, piled up, scattered, spread out, grouped, arranged in rows, and compared help to give primary children an understanding of concepts of quantity at the same time they are learning the numerals which represent quantity. Counting aids should be readily available to students for formal classroom activity and also for individual experimentation and investigation. In addition to the many things which children naturally collect, particularly useful devices include counting frames, blocks, discs, beads, toy animals, vehicles, and figures of all kinds. Some of these are available in counting kits which are useful in story-telling problems. For example, a toy farm can be used to illustrate the relationship between the number of cows and the number of stalls which are available. The symbols for the order relationships of more than (symbolized >) and less than (<) should be developed at the same time youngsters begin to count. A thorough understanding of quantity and the representation of quantity by the use of numerals can do much to build a firm foundation for the manipulation of symbols in algorisms.

PART II. OUR OWN NUMERATION SYSTEM

ORIGIN

The exact beginning of our numeration system is known only vaguely. Some historians believe that it has been in use since the thirtieth century B.C. Apparently it was employed for some time by the nomadic tribes of Arabia and India before it was recognized as being superior to the old Greek and Roman systems. The symbols are of Hindu origin and may have been picked up in India by Arabian traders and carried to Europe only as a curiosity. Hence our system of notation is called Hindu-Arabic. At first it appeared without the zero, and in this respect had little advantage over the older systems.

The zero may have come into use about the ninth century. The Hindus had hit upon the ingenious idea of place value, or grouping of numerals. By using a base of ten, they could express any amount with only ten symbols, using zero as the symbol to represent an empty place. The choice of all the symbols is arbitrary, as pointed out in Part I.

The Hindu-Arabic system was known in Europe as early as the 13th century but was not used extensively until developments in science and trade made its computational advantages preferred over the older existing systems, possibly about the 16th century. It is important to remember that over the past 5,000 years or so our numeration system has altered as it has been influenced by the changing needs of science and society and by the contributions of the pioneers in pure mathematics.

PLACE VALUE

Because we count in groups of numbers it is possible to represent quantity and to compute with only a few symbols. In our numeration system we count in groups of tens. We call this a base of ten. It is sometimes called a model group of ten. We have ten symbols: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and use positional notation in writing our numerals. We usually refer to this as place value. Zero enables us to indicate an empty position. The numeral 3 shows 3 ones. If we place the 3 in another position and write 30, we use the zero to occupy the ones' place and use the 3 in tens' place. It now represents 3 tens. Zero indicates an absence of ones. The numeral 30 represents 3 tens plus 0 ones. We can show the tens' place empty with another zero and place the 3 in the hundreds' place, giving 300. In each case, we have used the same symbol (3) to stand for three different amounts. We could go on doing this without end.

The numeral 10 represents ten only under certain conditions. We use a digit in a place to which we have assigned the value of ten and the zero indicates that there is nothing in the ones' place. We read this numeral, 10, as "ten." It represents one group of ten which is the base of the base ten numeration system. The symbol 1 in the numeral 10 represents 1 times the base of ten; and 2 in the numeral 24 represents 2 times the base of ten. If the base of the system were a different amount, the 1 in the numeral 10 would represent that amount. In considering the numeral 24, the 2 would represent twice the base or the model group of the system. Therefore, the numeral 10 represents which we call ten only when the base of the system is what we know as ten. Decem means ten. We have a "decimal" system of notation.

If we think about counting, it helps us to visualize the base of the system. It is perhaps unfortunate that we learn to count before we understand the meaning, because we are then so familiar with the process we may fail to see the importance of it. An understanding of the meaning of the base is necessary to the development of skillful computation.

We use counting numbers 1, 2, 3, 4, 5, 6 ... to represent quantities used in counting. These are called the natural numbers. Zero is not considered a natural number. The first number name in the ordered set of natural numbers used in counting is one. Zero is used to name the members of an empty set. Counting can be thought of as adding by ones. We think of cardinal numbers as the numbers indicating how many: one, two, three, and so on. The cardinal number 3 can be put into 1-1 relationship with /// things. When we count, the last number named states the name of the number of members of the set which we are counting. We count: "one, two, three..." "Three" is the last number we name. The term "third" names the member of a group that is between the second and fourth members. Ordinal terms such as first, second, third, fourth, etc., express the order or arrangement of a series of the members of a set of things.

In a numeral a digit has two values -- a cardinal value and a place value. The numeral 3 not only has a value that makes it equal to /// things; it also can be used to represent 3 tens, 3 hundreds, and so forth, according to its position in a numeral. We can think of a numeral as though the digits multiplied the value of the particular place in which each stands.

For instance, 333 can be expressed as $(3 \times 100) + (3\times10) + (3 \times 1)$. We call this an expanded form. We can express any numeral in this form: $25 = (2 \times 10) + (5 \times 1)$. It should be thoroughly understood that zero in the numeral 20 represents 0×1, indicating the absence of any members in the set of ones. In other words, the ones' place is empty. When a numeration system is thought of in this way it is possible to understand the arithmetic operations of addition, multiplication, subtraction, and division in any base to supply a foundation for the regrouping process which we often call "borrowing" and "carrying."

In order to demonstrate the value of each position in any numeral, students should develop a chart like the following by writing out selected sequences of numerals in expanded notation. (The ... indicates omissions in the illustrated sequence and the parentheses, multiplication.)

1 = 1(1)
2 = 2(1)
3 = 3(1)
4 = 4(1)
...
10 = 1(10) + 0(1)
11 = 1(10) + 1(1)
...
20 = 2(10) + 0(1)
...
26 = 2(10) + 6(1)
...
45 = 4(10) + 5(1)
46 = 4(10) + 6(1)
...
99 = 9(10) + 9(1)
100 = 1(100) + 0(10) + 0(1)
...
150 = 1(100) + 5(10) + 0(1)
...
268 = 2(100) + 6(10) + 8(1)
...
1000 = 1(1000) + 0(100) + 0(10) + 0(1)
...
1004 = 1(1000) + 0(100) + 0(10) + 4(1)
...
4579 = 4(1000) + 5(100) + 7(10) + 9(1)
...

THE ABACUS (plural, abaci (sī), - cuses)

The abacus has been used as a computational device for thousands of years. The columns of the various types of abaci serve to emphasize the close relationship between positional notation and the algorisms used in computation. They also make it possible to represent physically the regrouping necessary in computing and to demonstrate the use of a base in our numeration system. The abacus is a valuable tool for the development of an understanding of counting, place value, and regrouping. Students should not spend

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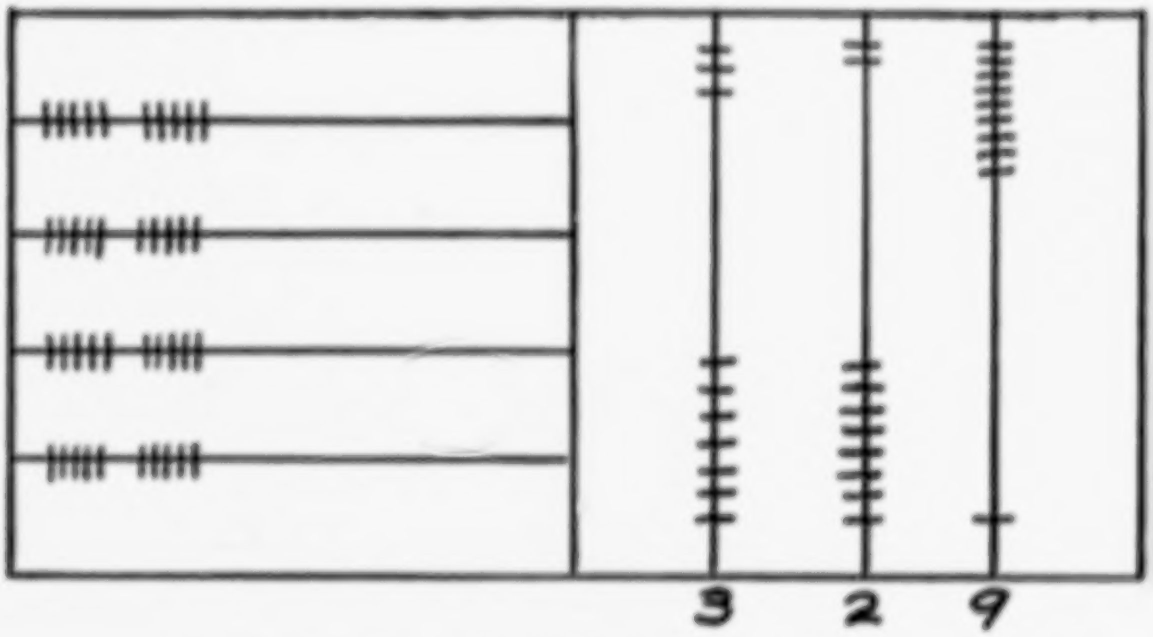
time needed to perfect techniques of computing with the abacus, since it will have little practical use for them. It can be very useful in the development of mathematical understandings.

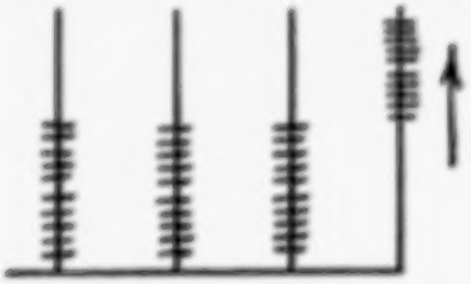
There are several types of abaci available. Some, combined with counting frames, are usually of a limited numerical capacity but may be used to build simple understandings. Place value is of course limited to the number of columns on the device. The abacus below represents the numeral 329.



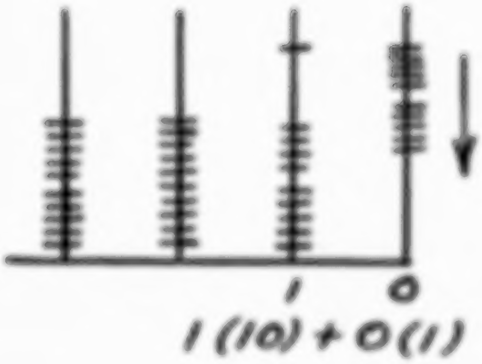
3(100) + 2(10) + 9(1)

Abaci of this type are usually equipped with 9, 10,or 20 beads on a column. The 20-bead abacus permits greater freedom in the manipulations involved in demonstrations of regrouping. The counting frame on the left may be used as described previously.

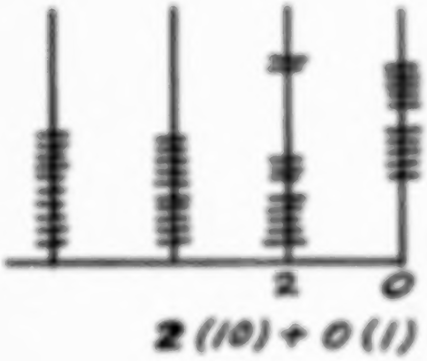
An abacus can be used to demonstrate place value as it relates to the model group of ten in our numeration system and to emphasize the quantity represented by each symbol in a numeral.



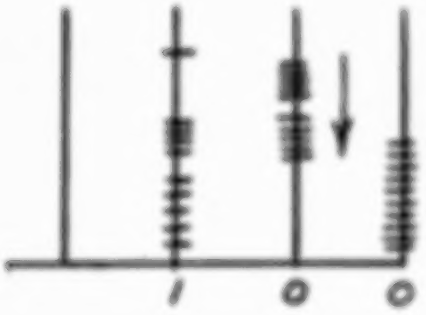
Push up 10 ones on the ones' column, counting and writing the numerals as you do: "1, 2, 3, 4, 5, 6, 7, 8, 9 ..." We have no single digit symbol for ten such as the Egyptian ∩, Roman X, and Greek ι .



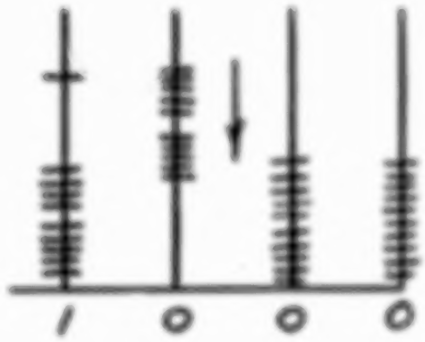
After writing 9 we have no more symbols to express additional quantities--we use place value and write 10, indicating 1 x 10 plus 0 x 1. Push down the 10 ones beads and push up 1 ten bead on the tens' column, demonstrating an exchange of 10 ones for 1 ten. Point out that the ones' column is empty and that this is indicated by the 0 in the numeral 10.



Do this again: push up 10 ones, count and write the numerals as you do: 11 12, 13, 14, 15...20. There is now another group of 10 ones, so push down the ones and exchange them for 1 ten bead on the tens' column. There are now two groups of ten represented by the numeral 20. Point out that this means (2 x 10) + (0 x 1).

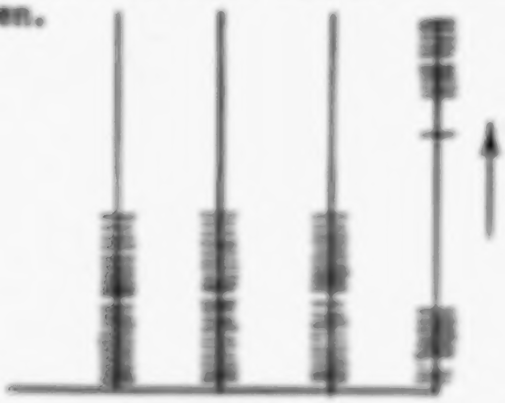


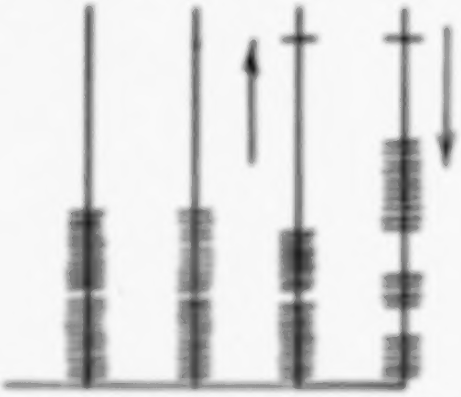
Continue doing this until 10 of the tens beads have been pushed up. Push the 10 tens beads down and exchange them for 1 hundreds bead. Point out that the numeral 100 represents (1 x 100) + (0 x 10) + (0 x 1). Show that the tens' column and the ones' column are both empty. This is indicated by the zeros in the numeral 100.



This procedure could be continued (or shortened) to demonstrate that after all of the hundreds beads have been used we need a single digit to represent 10 tens, or we can use place value. The Greek, Egyptian, and Roman symbols did this, ours do not. Exchange 10 hundreds for 1 thousands bead. Write the numeral 1000, indicating (1 x 1000) + (0 x 100) + (0 x 10) + (0 x 1). Point out that the hundreds', tens', and ones' columns are empty. This is indicated by the zeros in the numeral 1,000.

Use the 20-bead abacus to demonstrate similar numbers to primary children.



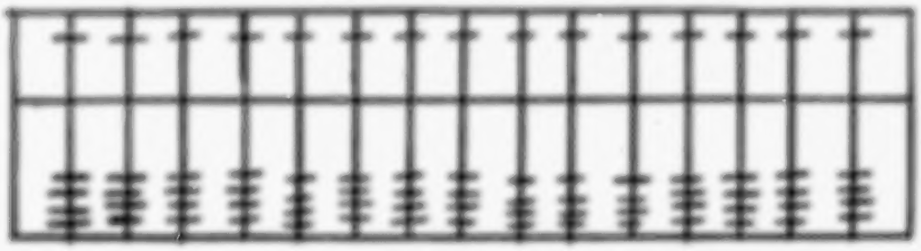


Encourage students to participate and to experiment. Let one child represent eighteen ones on the ones' column. Ask, "Who can regroup eighteen ones as tens and ones and write the numeral on the chalkboard?" Continue with other examples.

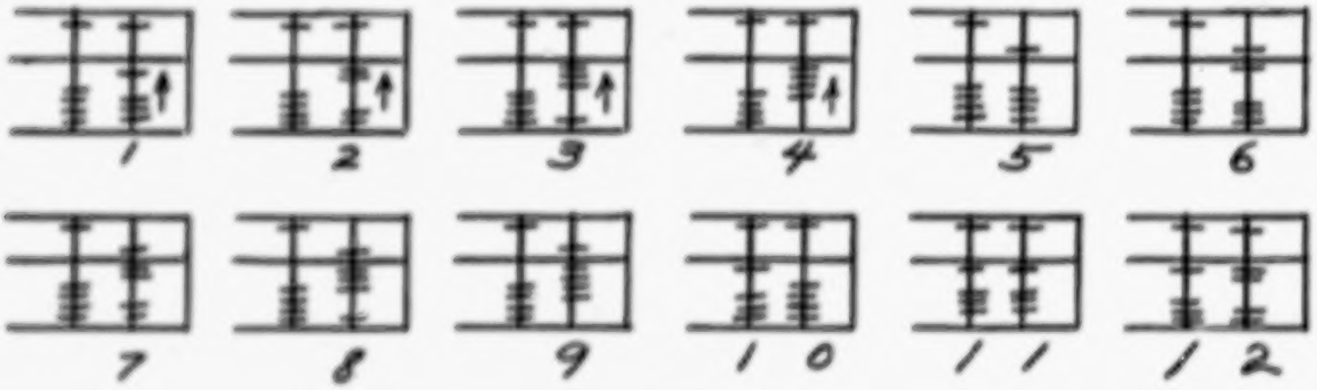
Adjust the size of the numbers to the abilities of the youngsters; let them make assumptions, and perform the manipulations. Primary children should be able to understand that eleven is equivalent to 11 ones, and also that there are 1 ten plus 1 one in 11, and similar examples.

THE SOROBAN

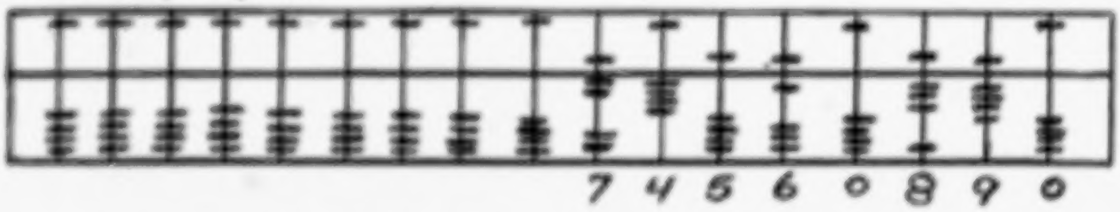
The Japanese abacus called the soroban uses columns which are divided by a cross bar. Each bead above the bar represents 5, each bead below the bar, 1. The beads are moved toward the cross bar in order to be "counted."



The diagrams below illustrate the position of the beads for counting from 1 to 12 on the soroban.



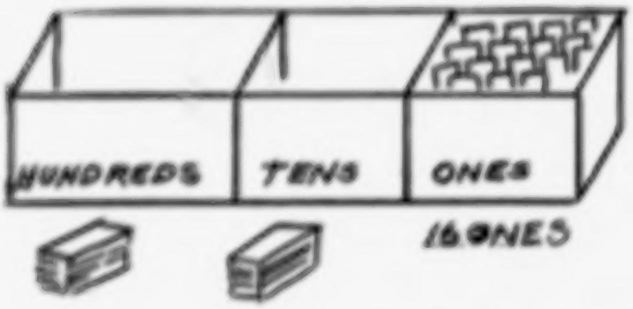
It is easy to see that very large numerals may be represented on the soroban. The beads against the bar in this illustration represent 74,560,890.

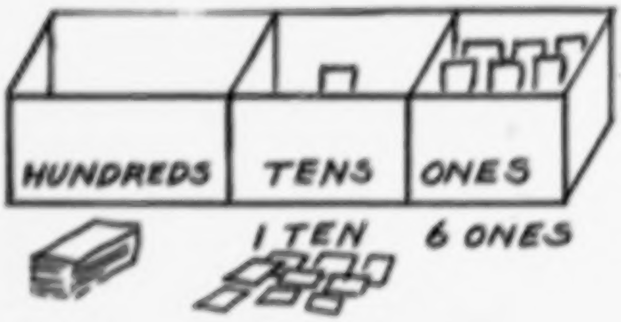


The numeral 74,560,890 may be expanded as (7 x 10,000,000) + (4 x 1,000,000) + (5 x 100,000) + (6 x 10,000) + (0 x 1,000) + (8 x 100) + (9 x 10) + (0 x 1). The place value is indicated by the columns on the soroban.

PLACE VALUE POCKETS

Use place value pockets in primary grades to demonstrate similar place value concepts in positional notation. Ask such questions as: "Who can regroup sixteen ones into 1 ten and 6 ones?"





Give the students cards representing sixteen ones or place sixteen ones in the ones' pocket. Have a student remove ten ones and exchange them for 1 ten. Place the 1 ten in tens' pocket. Ask that the numeral be written on the chalkboard.

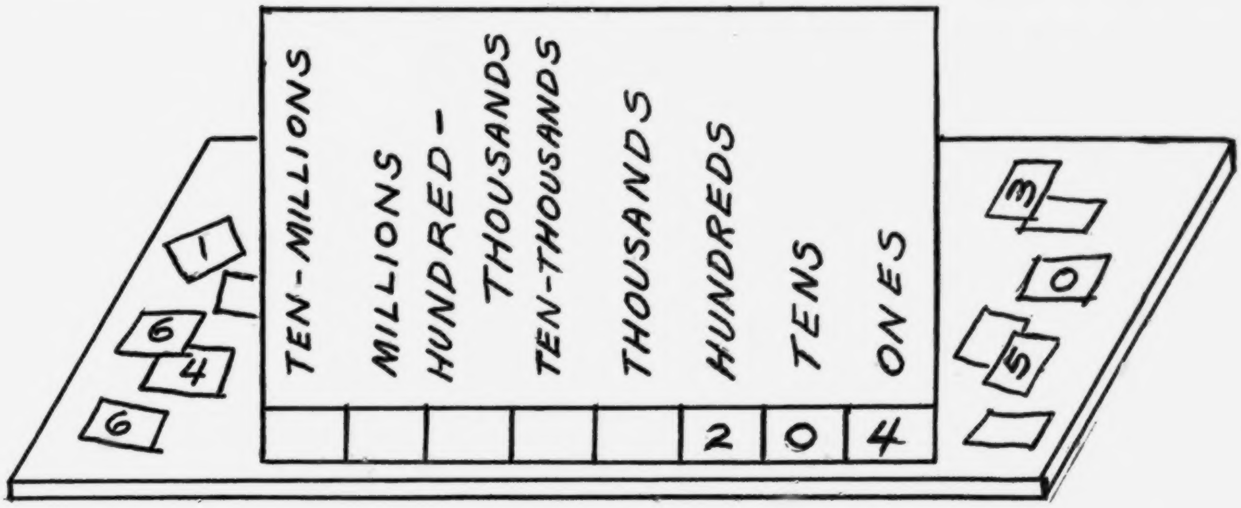
Discuss what the numeral represents.

16 = 16 ones
= 1 ten + 6 ones or 10 + 6
= (1 x 10) + (6 x 1)

Remove the 1 ten, and exchange it for 10 ones. Place the 10 ones back in the ones' pocket. Ask, "How many ones are now in the ones' pocket?" (16) "Do we still have the same amount (or quantity) as before we exchanged 1 ten for 10 ones?" (Yes)

PLACE VALUE CHARTS

A chart can also be used to demonstrate place value and to help give meaning to names which have to be memorized for positional notation.



Make small cards with the digits 0 to 9 on them. Place these as suggested in the illustration. Ask children to do such things as:

1. Place the 2,4, and 0 cards so that tens' place is empty and there are 2 hundreds and 4 ones represented. What is the numeral? (204)

2. Represent the numeral that contains 326 ones. What is the numeral? (326) How many tens does it contain? (32 tens or 32 and .6 tens) Can we say that it contains 32 tens and 6 ones? (Yes) Does it contain 3 hundreds 2 tens, and 6 ones? (Yes) Does(32 x 10) + (6 x 1) = (3 x 100) + (2 x 10) + (6 x 1)? (Yes)

3. Let students compute and discuss the above questions and similar ones in order to reinforce their understanding.

Use the place value chart to show that numbers may be thought of in ways such as:

4569 = 4569 ones
= 45 hundreds + 69 ones
= 456 tens + 9 ones or 456.9 tens
= 4 thousand + 5 hundred + 6 tens + 9 ones
= (4 x 1,000) + (5 x 100) + (6 x 10) + (9 x 1)
= 4.569 thousands

Represent these and other numbers on the place value charts. Use quantities suited to children's abilities. Ask questions that will lead students to come to their own conclusions. When they are in doubt, encourage experimentation and computation to help them make discoveries which will enrich their background and understanding.

millions	hundred-thousands	ten-thousands	thousands	hundreds	tens	ones
10 x 10 x 10 x 10 x 10 x 10	10 x 10 x 10 x 10 x 10	10 x 10 x 10 x 10	10 x 10 x 10	10 x 10	10 x 1	

Charts can also be used to show that each place has a value ten times the value of the place to the right of it. For example, 10 is equal to 10 x 1, 100 is equal to 10 x 10 and 1,000 is equal to 10 x 10 x 10.

Have students compute the value of each place or position so that they understand that the value is determined by the number of times ten is used as a factor.

Illustrate with exercises, as:

136 = 1(10 x 10) + 3(10) + 6(1)
4389 = 4(10 x 10 x 10) + 3(10 x 10) + 8(10) + 9(1)

When students understand the value of each position, have them complete computations such as:

429 = 4(10 x ?) + 2(?) + 9(?)
3263 = 3(10 x ? x ?) + 2(? x ?) + 6(?) + 3(?)

Refer to place value charts to find the missing quantities. Formulate exercises of increasing difficulty.

ten-thousands 1/10 of 100,000	thousands 1/10 of 10,000	hundreds 1/10 of 1,000	tens 1/10 of 100	ones 1/10 of 10

Each place also has a value 1/10 the value of the place to the left. For example, 10 is equal to 1/10 of 100, 100 is equal to 1/10 of 1,000, and 1,000 is equal to 1/10 of 10,000. Demonstrate this with computation or manipulations as suggested with place value pockets. Use whatever method is suited to the age and ability of children. A chart of this type can be further extended to include the use of decimal fractions. One-tenth is equal to 1/10 of 1, 1/100 is equal to 1/10 of 1/10, 1/1000 is equal to 1/10 of 1/100, etc. Decimal notation is shown below. Students should compute the value of each position to understand the manner in which the value of each digit in a numeral is determined in decimal notation. The numeral below represents 6(10,000) + 3(1,000) + 4(100) + 5(10) + 7(1) + 2(1/10) + 0(1/100) + 3(1/1000). Compute: 2/10 + 0/100 + 3/1000 to see that it may be expressed as .203 or two hundred three thousandths.

Students should do this.

ten-thousands 1/10 of 100,000	thousands 1/10 of 10,000	hundreds 1/10 of 1,000	tens 1/10 of 100	ones 1/10 of 10	tenths 1/10 of 1	hundredths 1/10 of 1/100	thousandths 1/10 of 1/100
6	3	4	5	7.	2	0	3

EXPONENT FORM

The value of each position in a numeral may be indicated with a simplified notation by using small numerals written above and to the right of the base, showing the number of times the base is used as a factor. We call these small numerals exponents. First consider our base ten system. In the example 10 x 10 = 100, 10 has been used as a factor twice. The quantity 100 can be represented by 10^2. The numeral 10^2 refers to the power. We read 10^2 as "ten to the second power," or as "ten squared." The expression 10^3 (meaning 10 x 10 x 10) is read "ten cubed" or "ten to the third power." Expressions such as 10^5(meaning 10 x 10 x 10 x 10 x 10) are read simply as "ten to the fifth" and so forth. It is very easy to see that the expression 10^8 is easier to read and write than 100,000,000 or the written words "one hundred million."

It is now possible to express the value of each position in a numeral by using exponential notation. The chart below shows how this may be done. Notice that the exponent names the number of zeros in each place or position.

ten millions 10 x 10 x 10 x 10 x 10 x 10 x 10	millions 10 x 10 x 10 x 10 x 10 x 10	hundred-thousands 10 x 10 x 10 x 10 x 10	ten-thousands 10 x 10 x 10 x 10	thousands 10 x 10 x 10	hundreds 10 x 10	tens 10 x 1	ones
10^7	10^6	10^5	10^4	10^3	10^2	10^1	10^0

We can see why ten to the zero power (10^0) may be used to express one if we do a problem using the powers of ten. We will start with an example to which we know the answer.

$$\begin{array}{r} 10 \\ \underline{\times\ 10} \\ 100 \end{array} \qquad \text{written in exponent form} \qquad \begin{array}{r} 10^1 \\ \underline{\times\ 10^1} \\ 10^2 \end{array}$$

As the product of 10 x 10 is 100, and 100 is expressed as 10^2 exponentially, the product of $10^1 \times 10^1$ would be expressed as 10^2 . In other words, $10^1 \times 10^1 = 10^{1+1}$ or 10^2. To multiply exponential expressions we add the exponents.

Now a simple division example will show why we may express ones as 10^0. Again we will use an example to which we know the answer, and try to discover how the exponential expressions may be used.

$$\begin{array}{r} 10 \\ 10\overline{)\,100} \end{array} \qquad \text{written in exponent form} \qquad \begin{array}{r} 10^1 \\ 10^1\overline{)\,10^2} \end{array}$$

This may also be expressed as:

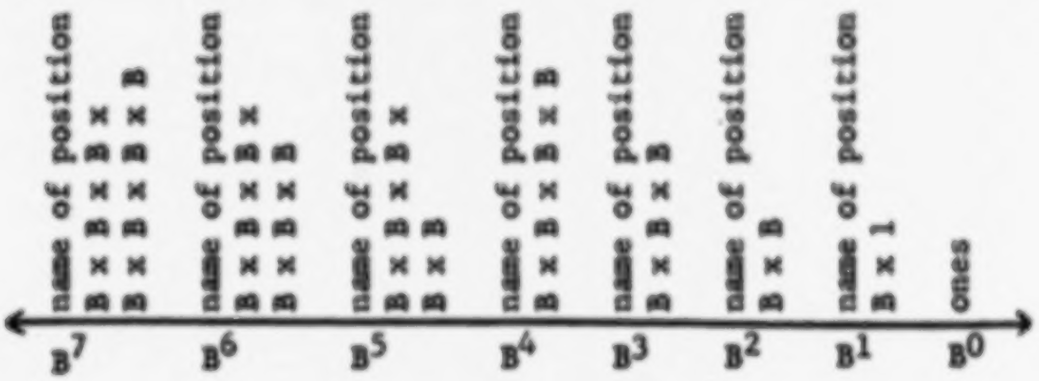
$$\frac{10}{10} = 1 \qquad \text{written in exponent form} \qquad \frac{10^1}{10^1} = 10^0$$

From the above examples it can be seen that we subtract exponents to divide exponential expressions. Therefore, $10^1 \div 10^1 = 10^{1-1}$ or 10^0. This would be true of any number which was used as the base. For example $3 \div 3 = 1$, or, in exponent form, $3^1 \div 3^1 = 3^{1-1}$ or 3^0. Using N to represent any number (N not 0), we can say that $N^0 = 1$, $10^0 = 1$, $3^0 = 1$, $45^0 = 1$, and so forth.

At this point we should again recall that our numeration system is based on a base of ten. Each position in a numeral can be expressed as a power of the base. The power is indicated with an exponent. Knowing these things we can (1) write any quantity that we wish, and (2) write the quantity in a system of numeration of any base we wish.

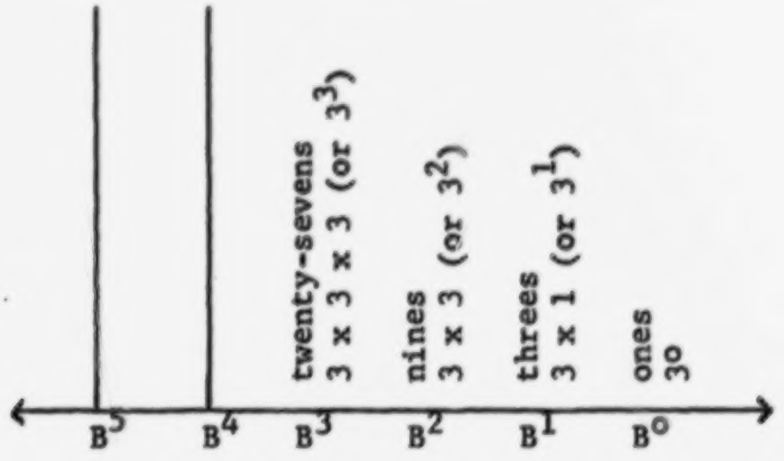
The place value chart may be generalized by using the letter B to represent any base as in the chart below. A chart like this gives a pattern of determining the value of each place or position in any system of numeration which uses place value as powers of a base.

ANY BASE


name of position
B x B x B x
B x B x B x B
name of position
B x B x B x
B x B x B
name of position
B x B x B x
B x B
name of position
B x B x B x B
name of position
B x B x B
name of position
B x B
name of position
B x 1
ones
B^7
B^6
B^5
B^4
B^3
B^2
B^1
B^0

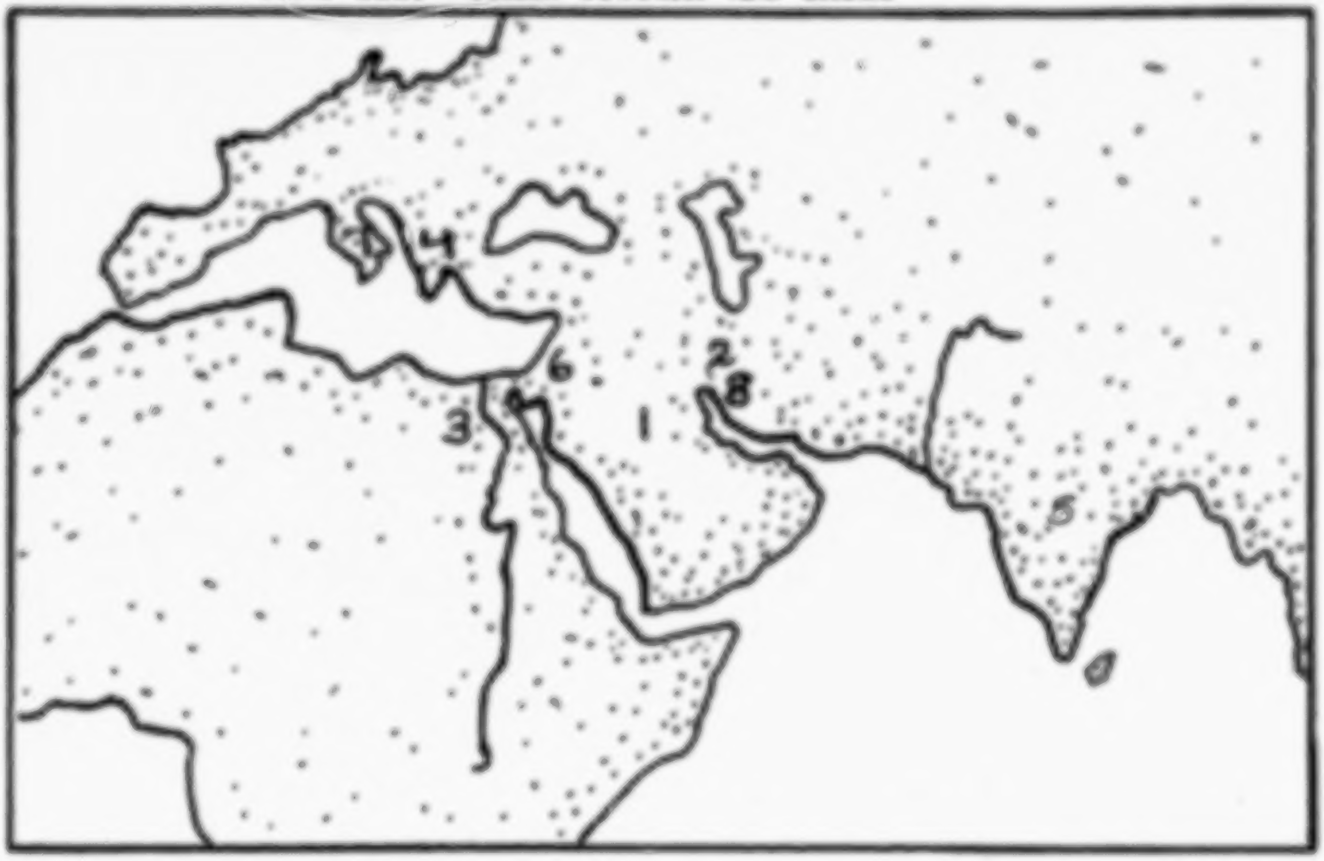
The names for each position, such as thousand, etc., could not be used for a base other than ten without confusion. Usually these are expressed in other bases in terms of the value which they would have in base ten. For example, in a base of three the position occupied by B^2 would be equivalent to 3 x 3 (or the B x B), which would have a value of nine in base ten. In a base three system of notation, the base ten values of the first three positions would be: ones, threes, and nines. Therefore, the numeral 100 in base three would be written in expanded form as 1(0) + 0(3) + 0(1), indicating a value of only 9 in base ten.

BASE OF THREE


twenty-sevens
3 x 3 x 3 (or 3^3)
nines
3 x 3 (or 3^2)
threes
3 x 1 (or 3^1)
ones
3o
B^5
B^4
B^3
B^2
B^1
B^o

It is evident from the foregoing considerations that memorization of the names for place value in base ten offers no real understanding of the fundamental properties of a system of numeration which employs a model group and positional notation. It is essential that place value be understood in terms of the powers of a base. It is not enough to teach students that we count and compute in groups of ten. Skill in computing requires students to understand the make-up of the system.

PART III. OTHER SYSTEMS AND BASES



1. Arabian
2. Babylonian
3. Egyptian
4. Greek
5. Hindu
6. Phoenician
7. Roman
8. Sumerian

THE BEGINNINGS OF MATHEMATICS

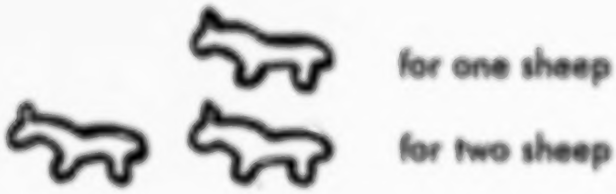
We believe that the first mathematical systems were produced by ancient civilizations clustered around the Mediterranean Sea and the Indian Ocean. Archeological remains give us some idea as to how people living as long as 6,000 years ago did their computing. Surprisingly, they had many advanced ideas about mathematics.

The use of maps and globes to identify the places of origin of various numeration systems would be very appropriate at this time. A time line extending to about 6,000 B.C. could be used to enhance the study of centuries by relating it to the development of mathematics. A similarly designed time line illustrating placement of major historic developments adds much interest. Important dates in economic and social development should be included, indicating the close relationship between cultural progress and the mathematics needed. Investigations might consist of a comparison of distances traveled in ancient Egypt with those traveled today and the numerical representations needed to record them. A similar comparison could be made between the units of measure used in ancient times and the more precise and standardized measures needed today.

THE EGYPTIAN NUMERATION SYSTEM

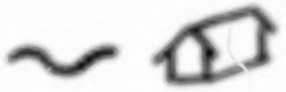
Egypt is one of the oldest civilizations of which we have a written record. From its temples and tombs, constructed some four to six thousand years ago, we have been able to learn a great deal about the life of the people and their mathematics. They seem to have been a clever folk, quick to invent mathematical methods that would aid them in their daily pursuits. The Egyptians, Sumerians, Babylonians, Arabians, Hindus, Greeks, and Romans have contributed to our mathematical system.

The Egyptians and the people of other countries in this area used a picture writing called hieroglyphics for words and for the quantities that they wished to represent. The earliest of these hieroglyphics were very simple. Man appears to have done his computing at first by drawing simple pictures to represent quantities. For instance:

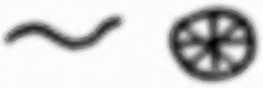


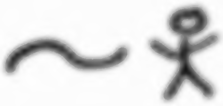
It is easy to see how something like this could lead to a hieroglyphic writing and numeration system. There is some indication that men once wrote words for each number. Instead of making a symbol such as 6 or 7, they wrote a word meaning six or seven. It would be very difficult to do much computing with this kind of a system. Symbols were probably invented as a matter of convenience.

One of the oldest of all these systems of symbols, the Egyptian, is quite remarkable for its simplicity and usefulness. The Egyptian numerals were written in hieroglyphics. There are words in our language which could be expressed in that manner. Watershed, water wheel, water snake, and the proper name Waterman might be written in hieroglyphics:

water shed 

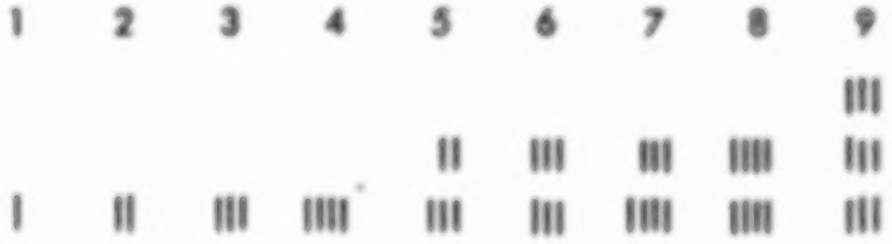
water snake

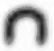
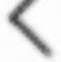
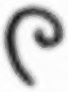
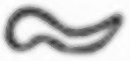
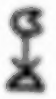
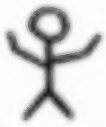
water wheel 

Waterman 

An understanding of hieroglyphics could be developed for primary children, using flannelboard cutouts. Intermediate grade students might investigate different hieroglyphic systems or produce their own, using clay and primitive handmade tools.

The Egyptians simply made marks or slashes for the numbers 1 through 9. Marks are easier to read when grouped. This is how they might have appeared:



10		heel bone	10,000		bent line
100		coil of rope	100,000		burbot (fish)
1,000		lotus flower	1,000,000		man in astonishment

We do not know why they chose these particular symbols. One might imagine that a number like a million would be represented as a "man in astonishment" who was simply overcome by such a large number. This reminds us of the very simple number system: "one, two, some, and many.

Show primary children the Egyptian system of using slashes to represent quantities from one through nine in a 1-1 relationship. Describe the kinds of materials used for computing in a land which had no paper--until the invention of papyrus. Encourage youngsters to use materials other than paper and pencil, such as clay, soft stone, and sticks. Point out that /// is more easily read as "six" than //////
///
because the slashes are grouped. Ask youngsters for suggestions as to the best way to"think" of each quantity. Even in this simple system and with crude materials there is a certain aesthetic appeal in writing numerals that are easily read.

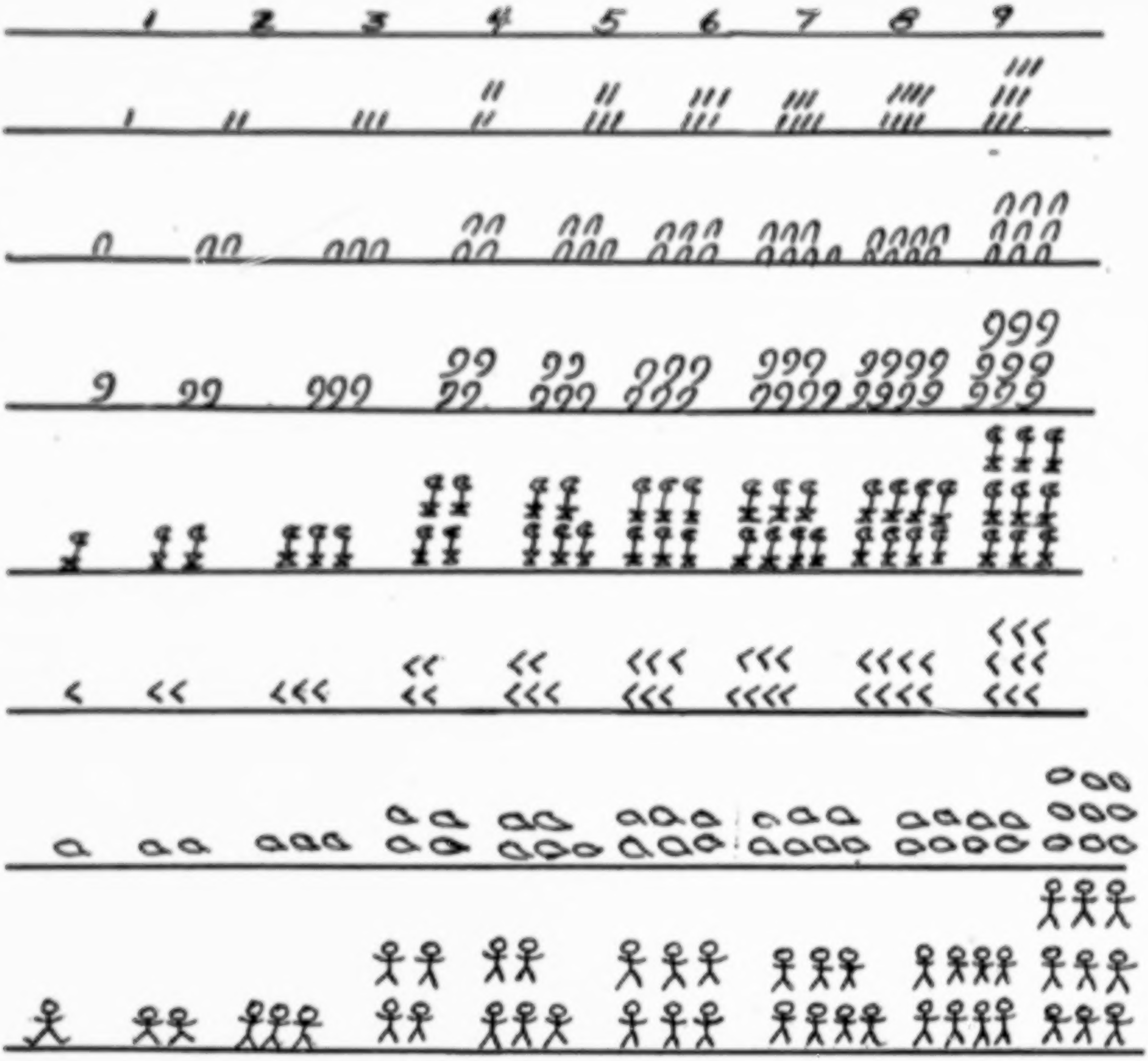
The formation of the numerals is demonstrated in the following chart. Present as much as is suited to the students' abilities. It should be noted that there are many different ways to group the symbols because there is no place value. For example, 500 could be written 𓍢𓍢 𓍢𓍢𓍢 , or 𓍢𓍢𓍢 𓍢𓍢, or 𓍢𓍢𓍢𓍢𓍢 (Chart on page 27).

A chart similar to the one below can be developed to help children understand our Hindu-Arabic system of numeration.

ones	1	2	3	4	5	6	7	8	9
tens	10	20	30	40	50	60	70	80	90
hundreds	100	200	300	400	500	600	700	800	900
thousands	1000	2000	3000	4000	5000	6000	7000	8000	9000

Point out to students that after we have represented the counting numbers through nine with numerals, we have used all of the symbols which we need in a base of ten. We then symbolize the group of ten as 10 which indicates 1 ten and 0 ones. Counting again we combine all the symbols in order, writing 10, 11, 12, 13, 14, 15, 16, 17, 18, 19,... We again have counted to the next group of ten. We have accumulated two groups of ten. We write it as 20, which indicates 2 tens plus 0 ones.

(Chart from Page 26)



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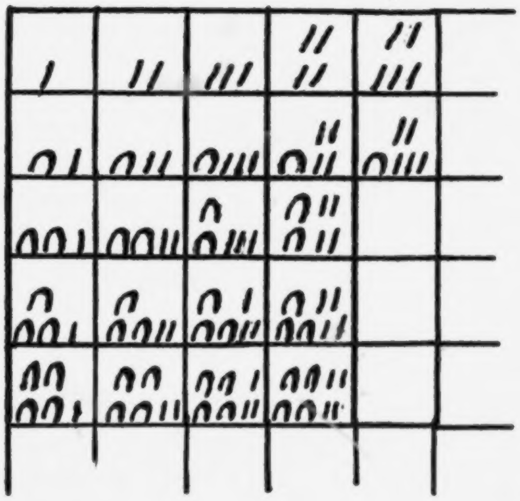
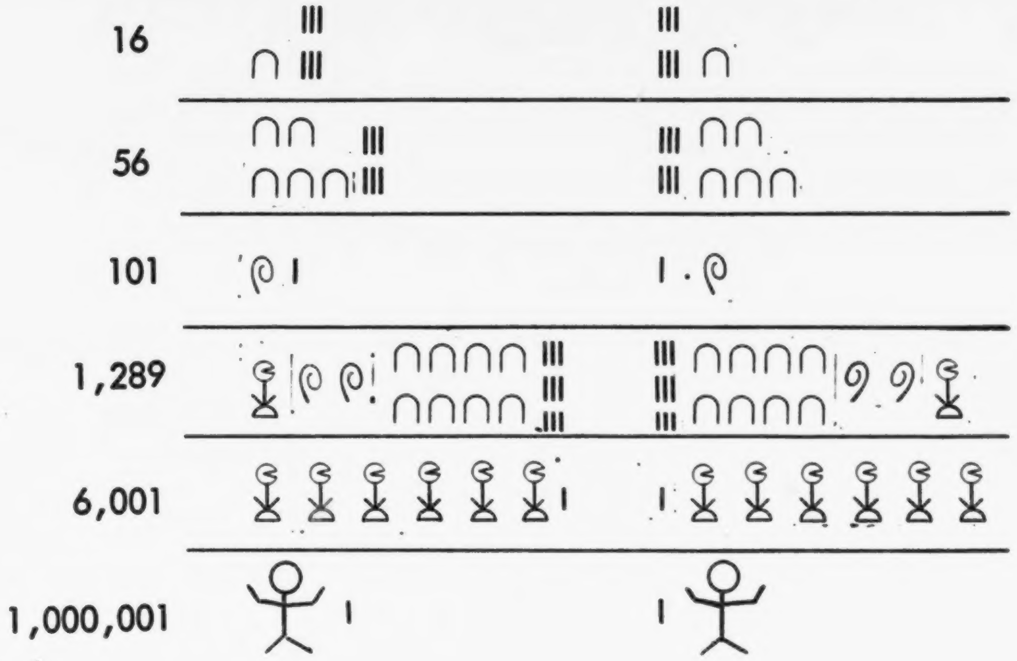
The hundreds board could be introduced at this time in connection with the Egyptian system. Intermediate-grade youngsters could make portions of a hundreds board in the Egyptian system and make comparisons with the Hindu-Arabic. It should soon become apparent that the Egyptian system required very long numerals. In addition to acquiring historical background in the development of mathematics, students could quickly identify the advantages of the base ten system. A suggested comparison is shown below.

1	2	3	4	5	
11	12	13	14	15	
21	22	23	24		
31	32	33	34		
41	42	43	44		



Encourage students to investigate the relationship between the materials available to the early Egyptians and possible methods of computation. Point out that the numerals were made in clay or chipped in sandstone. Let students invent algorisms and do simple computation in this fashion. Such a lesson points out the close relationship between the culture and the mathematics of the time. Clay and soft stone may be purchased from art suppliers.

The Egyptians had no zero. They had no need of it in this system. The zero was to come many years later. If we write a few Egyptian numerals, this will be evident.



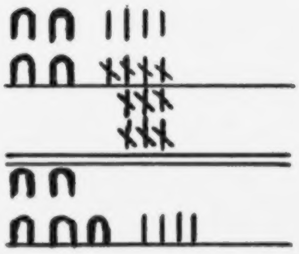
Since the Egyptians did not use place value, it would have made no difference to them if 122 were written as 𓍢𓎆𓎆𓏺𓏺, or 𓎆𓎆𓍢𓏺𓏺, or 𓏺𓏺𓍢𓎆𓎆. These numerals are written with the principle of addition: 100 + 10 + 10 + 1 + 1, 10 + 10 + 100 + 1 + 1, and 1 + 1 + 100 + 10 + 10 all equal 122. Accordingly, it is rather easy to compute in Egyptian hieroglyphics because all we have to do is to count the number of times a symbol appears, and regroup when necessary.

ADDITION IN HIEROGLYPHICS

A few examples will show a manner of regrouping in the operation of addition in the Egyptian system. The algorism or arrangement of the symbols in a numeral does not make any difference; it will be easier, however, to keep them in the order of a place value to which we are accustomed.

1.

$$\begin{array}{r} 48 \\ +6 \\ \hline 54 \end{array}$$

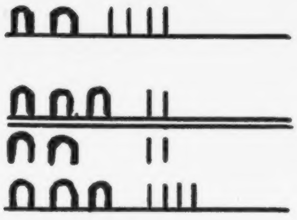


5(10) + 4(1) = 54

The total of 14 ones and 4 tens may be regrouped and written as 5 tens + 4 ones.

2.

$$\begin{array}{r} 24 \\ +32 \\ \hline 56 \end{array}$$



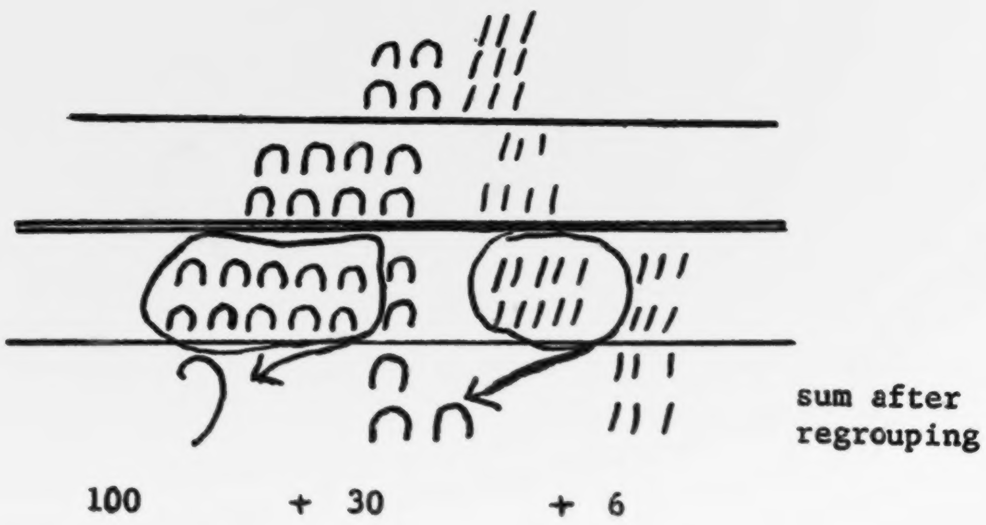
5(10) + 6(1) = 56

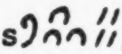
No regrouping is needed in this example. It is only necessary to copy the symbols in the addends into the sum.

The Egyptian system is like ours in that if we have enough tens, hundreds, etc., we "carry" (regroup) them in the next place. Doing examples like these helps to make this clear. When students use the Egyptian system, it might help to cross off the symbols that are regrouped in order to avoid counting the symbols twice. Students should be encouraged to experiment and to invent suitable algorisms for computing.

The following example suggests algorisms which might be used for more difficult computation.

$$\begin{array}{r} 49 \\ +87 \\ \hline 136 \end{array}$$



Count the tens and ones and write them down, as 12 tens and 16 ones (this is the first sum). Rewrite the 12 tens as 1 hundred and 2 tens. Rewrite the 16 ones as 1 ten and 6 ones (this is the second sum). The final sum is  (100 + 30 + 6 = 136). It would also be possible to regroup enough ones to make a ten and enough tens to make a hundred and write them down in the first place. Exercises like that above furnish a visual example of the development of algorisms for computation and methods of regrouping in positional notation. A longer addition example would be done in the same way as this one. Encourage students to develop their own methods for computing and let the class discuss advantages and disadvantages.

THE BABYLONIAN SYSTEM

The Babylonian system is of special interest since it relates to our recording of time, which can be computed in a base of sixty. Such computation might be illustrated as follows:

			1 (regroup	60 minutes as 1 hour)
1 hour	45 minutes		1 hour	45 minutes
+ 3 hours	50 minutes	OR	+ 3 hours	50 minutes
4 hours	95 minutes		5 hours	35 minutes
+ 1 hour	-60 minutes			
5 hours	35 minutes			

The Babylonians, whose symbols were imprinted in wet clay with a stylus, represented ten as ◁ and one as ▽. By using place value and later developing a symbol to represent an empty place, they could write any amount. This system is based on a model group of sixty. The chart below shows a comparison of base ten and base sixty.

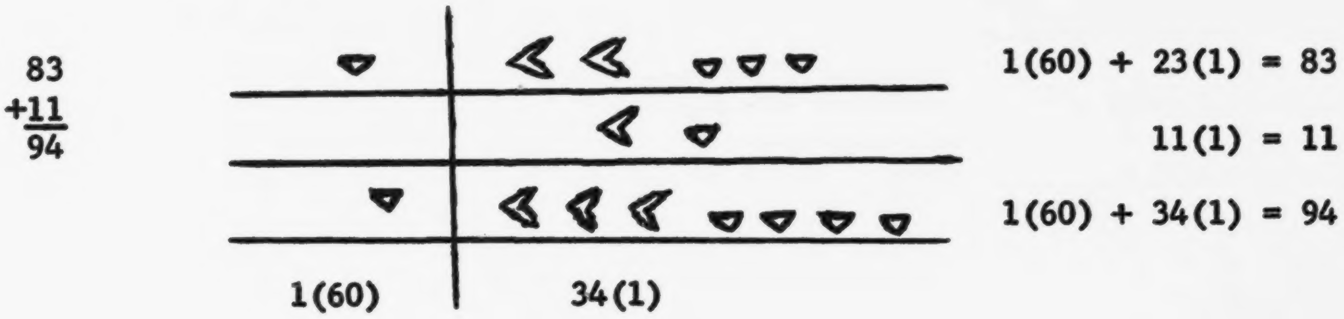
10x10x10 (10^3)	10x10 (10^2)	10x1 (10^1)	(10^0)
1,000's	100's	10's	1's
60x60x60 (60^3)	60x60 (60^2)	60x1 (60^1)	(60^0)
216,000's	3,600's	60's	1's

Place value is indicated by writing numerals in columns. The columns in this system separate the numerals into periods in the same way as commas do in the Hindu-Arabic.

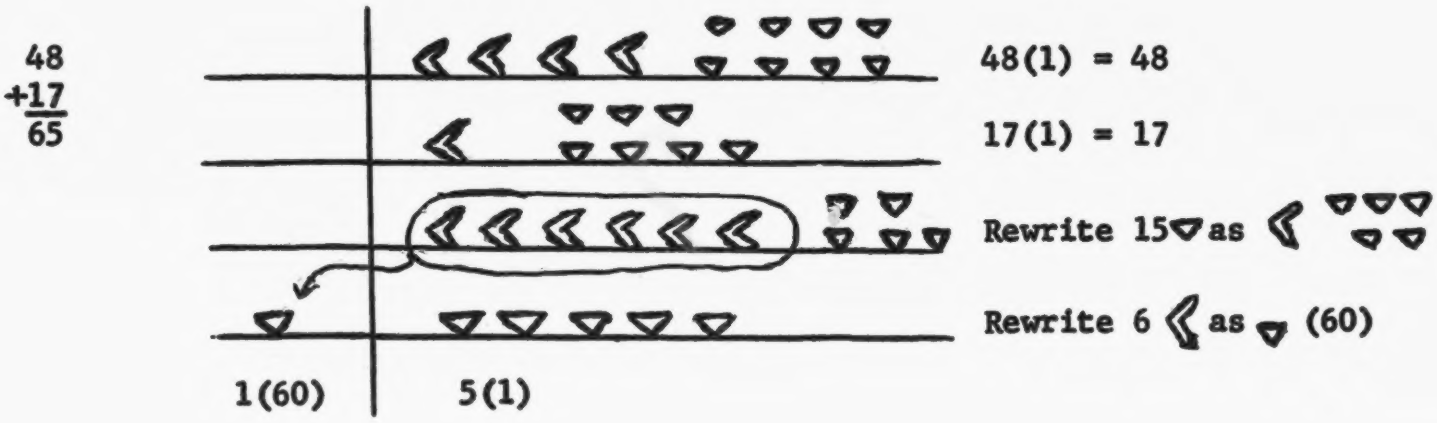
	216.000	3.600	60	
1				▽
36				◁◁◁ ▽▽▽▽▽▽
60			▽	
79			▽	◁ ▽▽▽▽▽▽▽▽▽
3,611		▽		◁ ▽

1 = 1(1)
36 = 3(10) + 6(1)
60 = 1(60)
79 = 1(60) + 1(10) + 9(1)
3,611 = 1(3,600) + 0(60) + 10(1) + 1(1)

As this system developed the columns were not always clearly shown. Empty columns sometimes were indicated by leaving a larger space between symbols, instead of using a symbol. Computation and translation therefore are difficult. There are many variations as in other numeration systems. The operations in our Hindu-Arabic system, however, can be demonstrated by doing simple exercises with the Babylonian symbols. As in the case of the Egyptian system, students are forced to indicate the regrouping necessary in order to "carry" or "borrow." The two simple exercises below will suggest some of the ways that these symbols could be manipulated in computation.



Notice that we are forced to think in groups of 60 and that the left column now includes quantities we usually represent as 2 digit numerals. Ones' column has the quantities from 1 to 59 in it, because the second position or column is that of 60.



A better understanding of the development of mathematics may be engendered when students investigate such systems as the Egyptian and Babylonian. Encourage students to experiment in originating algorisms for the operations, using only the materials which would have been available at the time the system was in use. Have supplies of clay, sandstone, small sticks, etc., for this purpose. The limitations and advantages of the systems might be compared. Students should understand that methods of computation and "keeping track" of things have grown and changed with the needs of the culture and the ingenuity of mathematicians.

THE GREEK SYSTEM

The ancient Greeks used the letters of their alphabet as symbols for numerals in their system of numeration. The principle of addition was applied, as in the Egyptian system, in giving value to the Greek numerals. Some of these appear below.

1	α	alpha	10	ι	iota	100	ρ	rho
2	β	beta	20	κ	kappa	200	σ	sigma
3	γ	gamma	30	λ	lambda	300	τ	tau

(Iota is pronounced yōta.)

Separate symbols were used for each position in a numeral. For example: The quantity 222 would be written σκβ, as 200 plus 20 plus 2, with a different symbol for each value of 2. It is obvious that this system of numeration would be very confusing when quantities were written in a narrative context. Various methods were devised so that the numerals could be distinguished from the letters. Perhaps this is one reason that the Greeks viewed mathematics as a recreation and spent a great deal of time on the development of geometry.

ROMAN NUMERALS

In order to write Roman numerals, we use principles of addition, subtraction, multiplication, and repetition. When this is understood, reading and writing Roman numerals becomes a simple process.

The numerals are written with the use of seven symbols: I, V, X, L, C, D, and M. This is a very old system of notation which we use today in chapter headings, sections of an outline, clock faces, and the names of space vehicles such as "Saturn V." Over the centuries there have been changes in the symbols as we now know them. At the time Roman numerals first appeared people probably did not use very large quantities in calculation. As the culture of the people developed and their possessions increased, they had need of a better system. The symbols were changed to fit particular needs. We are experiencing a similar need today in computing interplanetary distances and speed.

The Roman numeral V once was written as Λ ; M, the symbol for one thousand, was once ᗡD, or 2 five hundreds put together. There have been many other changes. A bar over a numeral multiplied it by 1,000. Twenty thousand could be written as 20 M's in a row or simply as $\overline{XX}$. One million would be $\overline{M}$. (1,000 x 1,000 = 1,000,000)

THE USE OF ADDITION

The Romans used a place value of sorts in that a smaller number after a larger number was added to the larger number. The following are examples of the use of addition.

VI	(5 + 1 = 6)
VIII	(5 + 1 + 1 + 1 = 8)
XII	(10 + 1 + 1 = 12)
MDCXX	(1000 + 500 + 100 + 10 + 10 = 1620)
LXXV	(50 + 10 + 10 + 5 = 75)
DCCCX	(500 + 100 + 100 + 100 + 10 = 810)

THE USE OF SUBTRACTION

With the exception of M (1000) and I (as in IIII on clock faces), no symbol is repeated more than three times. When it becomes necessary to repeat a symbol four times, a higher one is used and we subtract once. The underlined numerals use the principle of subtraction.

1	2	3	4	5	6	7	8	9	10
I	II	III	IV	V	VI	VII	VIII	IX	X

40	50	80	90	100	200	300	400	500	900	1,000
XL	L	LXXX	XC	C	CC	CCC	CD	D	CM	M

OTHER BASES

It has been shown that it is possible to symbolize any quantity with the use of only ten number symbols in a base ten system of notation. In the base ten system, "ten" is the model group. However, the base of a numeration system may be any size that is convenient. It is possible to have a numeration system based on some other group, such as 12, 7, 6, and so forth. We have used some of these as bases for a long time without realizing it.

We use a base of twelve in computing linear measurement. If the computation is thought of as being in a base of twelve, the operations of addition and subtraction are more understandable.

$$\begin{array}{rr}
2 \text{ feet} & 8 \text{ inches} \\
+\ 3 \text{ feet} & 6 \text{ inches} \\
\hline
5 \text{ feet} & 14 \text{ inches} \\
+\ 1 \text{ foot} & -12 \text{ inches} \\
\hline
6 \text{ feet} & 2 \text{ inches}
\end{array}$$

(removing 1 model group of 12 and regrouping it as 1 foot)

$$\begin{array}{rr}
5 \text{ feet} & 2 \text{ inches} \\
-\ 2 \text{ feet} & 6 \text{ inches} \\
\hline
4 \text{ feet} & 12 + 2 \text{ inches} \\
-\ 2 \text{ feet} & 6 \text{ inches} \\
\hline
2 \text{ feet} & 8 \text{ inches}
\end{array}$$

(regrouping 5 feet 2 inches as 4 feet 14 inches. Subtraction can now be completed)

Science and industry find uses for bases other than ten. It is sometimes desirable to organize a system of numeration to fit a particular situation. In order to be able to symbolize and to understand another system using a different base, it is well to learn first to count in the system. As has been pointed out previously, counting and the use of a base are basic to the operations of addition, multiplication, division, and subtraction. A further consideration of counting in the familiar base ten will aid in understanding the use of a base of seven.

COUNTING IN BASE SEVEN

The base of the system to be used dictates the number of symbols which will compose the numerals. We employ ten symbols to write the numerals for all the quantities in base ten: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and use positional notation with zero to indicate an empty position.

To do this we give digits two values: (1) An amount which the digit itself symbolizes (how many), and (2) an amount which it has due to its position in the numeral, indicating the number of groups it represents.

When we have counted through nine we use place value and write 10. We call it "ten." Ten is the name of the place which 1 occupies in the numeral 10. It is also the name of the size of the base of a base ten system of notation. The numeral 10 indicates that the base is taken 1 time and that there are 0 ones.

If, instead of counting in groups of ten, we can change the size of the group and count in a base of seven, then we have changed the value of the positions in the numerals in the system of notation. We have, and need, only seven digits, using zero as the numeral that stands for no objects to indicate an empty place. To understand the values of each position we need to recall that they are determined as powers of the base. In base ten we have ones; tens (10 x 1); hundreds (10 x 10); thousands (10 x 10 x 10); and so forth. In base seven we have ones; sevens (7 x 1); forty-nines (7 x 7); three-hundred-forty-threes (7 x 7 x 7); and so forth. The only difference is that we are accustomed to the names for the value of each position in the numerals in base ten. We have no such names as "hundred" (10 x 10) for the value of 7 x 7 in a base of seven. We can either make up new names for each position or use the name that indicates the value in a base of ten. Making up new names necessitates extra memory work, so it will be easier to think of each position in base seven as the name of its value in base ten.

The digit 7 cannot be used in writing the numerals in base seven, because seven is the base and is now written as 1 group of sevens and no ones, with the numeral 10. In order to avoid confusion with base ten notation (in which the second position has a value of ten), we will name the digits in the numeral when it is written in base seven. The numeral 10 will be called "one zero" or "one zero base seven." The value of the second place has now been changed to seven instead of ten. For example: Ten in base seven is written as 13 which indicates 1 seven plus 3 ones. The numeral 10 in base ten indicates 1 ten plus 0 ones. The base seven numeral 13 and the base ten numeral 10 both have a value of ten.

Numerals in other bases are indicated in two ways--as 13_{seven} or as 13_7. We do not write 13_{ten} when we know that the numeral is in base ten. Neither shall we do so when it is evident that the numeral is written in base seven. When the base is not evident it will be indicated with a subscript seven or the digit 7.

The two charts below illustrate the relation of the size of the base to the value which the numeral represents.

BASE TEN	BASE SEVEN
1, 2, 3, 4, 5, 6, 7, 8, 9, ...?	1, 2, 3, 4, 5, 6, ...?
10 = 1(ten) + 0(one)	10 = 1(seven) + 0(one)
11 = 1(ten) + 1(one)	11 = 1(seven) + 1(one)
12 = 1(ten) + 2(one)	12 = 1(seven) + 2(one)
13 = 1(ten) + 3(one)	13 = 1(seven) + 3(one)
....	....
19 = 1(ten) + 9(one)	16 = 1(seven) + 6(one)
20 = 2(ten) + 0(one)	20 = 2(seven) + 0(one)
21 = 2(ten) + 1(one)	21 = 2(seven) + 1(one)
....	....
99 = 9(ten) + 9(one)	66 = 6(seven) + 6(one)
100 = 1(one hundred) + 0(ten) + 0(one)	100 = 1(forty-nine) + 0(seven) + 0(one)

After we count to nine in base ten, the next number is ten, the base of the system. After we count to six in base seven, the next number is seven, the base of the system. In base ten, 9 + 1 = 10, written in expanded notation as 1(10) + 0(1). In base seven, 6 + 1 = 10, written in expanded notation as 1(7) + 0(1). This can be shown with the familiar regrouping process of addition.

BASE TEN

$$\begin{array}{r l} 1 & \text{(regroup 10 ones as 1 ten)} \\ 9 & \\ +\ 1 & \\ \hline 10 & 1(\text{ten}) + 0(\text{one}) \end{array}$$

BASE SEVEN

$$\begin{array}{r l} 1 & \text{(regroup 7 ones as 1 seven)} \\ 6 & \\ +\ 1 & \\ \hline 10 & 1(\text{seven}) + 0(\text{one}) \end{array}$$

A like regrouping takes place in the position of the base squared. In base ten, 99 + 1 = 100, which represents 1(one hundred) + 0(ten) + 0(one). In base seven, 66 + 1 = 100, which represents 1(forty-nine) + 0(seven) + 0(one).

$$\begin{array}{r l} 1 & \text{(10 tens regrouped as 1 hundred)} \\ 1 & \text{(10 ones regrouped as 1 ten)} \\ 99 & \\ +\ \ 1 & \\ \hline 100 & 1(\text{hundred}) + 0(\text{ten}) + 0(\text{one}) \end{array}$$

$$\begin{array}{r l} 1 & \text{(7 sevens regrouped as 1 forty-nine)} \\ 1 & \text{(7 ones regrouped as 1 seven)} \\ 66 & \\ +\ \ 1 & \\ \hline 100 & 1(\text{forty-nine}) + 0(\text{seven}) + 0(\text{one}) \end{array}$$

PLACE VALUE IN BASE SEVEN (The Abacus and Place Value Charts)

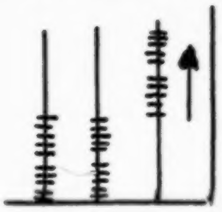
The following charts indicate the names which will be used for the positions in base seven numerals as compared to base ten.

ten thousands (10,000 = 10 x 1,000)	thousands (1,000 = 10 x 100)	hundreds (100 = 10 x 10)	tens (10 = 10 x 1)	ones
10^4	10^3	10^2	10^1	10^0

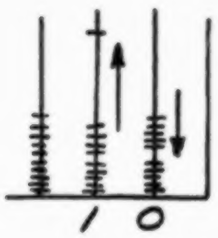
two-thousand-four hundred-ones (2,401 = 7 x 343)	three-hundred-forty-threes (343 = 7 x 49)	forty-nines (49 = 7 x 7)	sevens (7 = 7 x 1)	ones
7^4	7^3	7^2	7^1	7^0

Positional notation can be demonstrated with an abacus in base seven in the same manner as in base ten. Such a comparison will serve to emphasize the importance of the base in a numeration system as illustrated below.

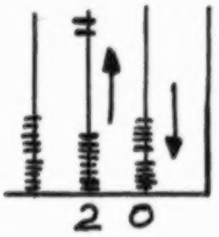
BASE TEN ABACUS



Push up ten ones in ones' column, counting and writing the numerals as you do: 1,2,3,4,5, 6,7,8,9,...?

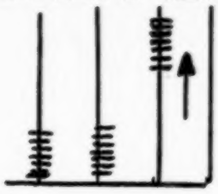


After 9 we have no more symbols to express additional quantities. Push down 10 ones, exchange for 1 ten bead. Write 10 = 1(10) + 0(1)

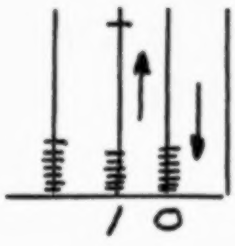


Repeat: push up ten more ones. Exchange 10 ones for 1 ten bead. Write 20 = 2(10) + 0(1).

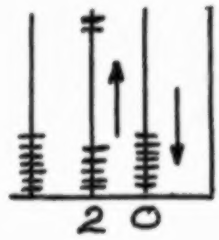
BASE SEVEN ABACUS



Push up 7 ones, on ones'column, counting and writing the numerals as you do: 1,2,3,4,5,6,...?



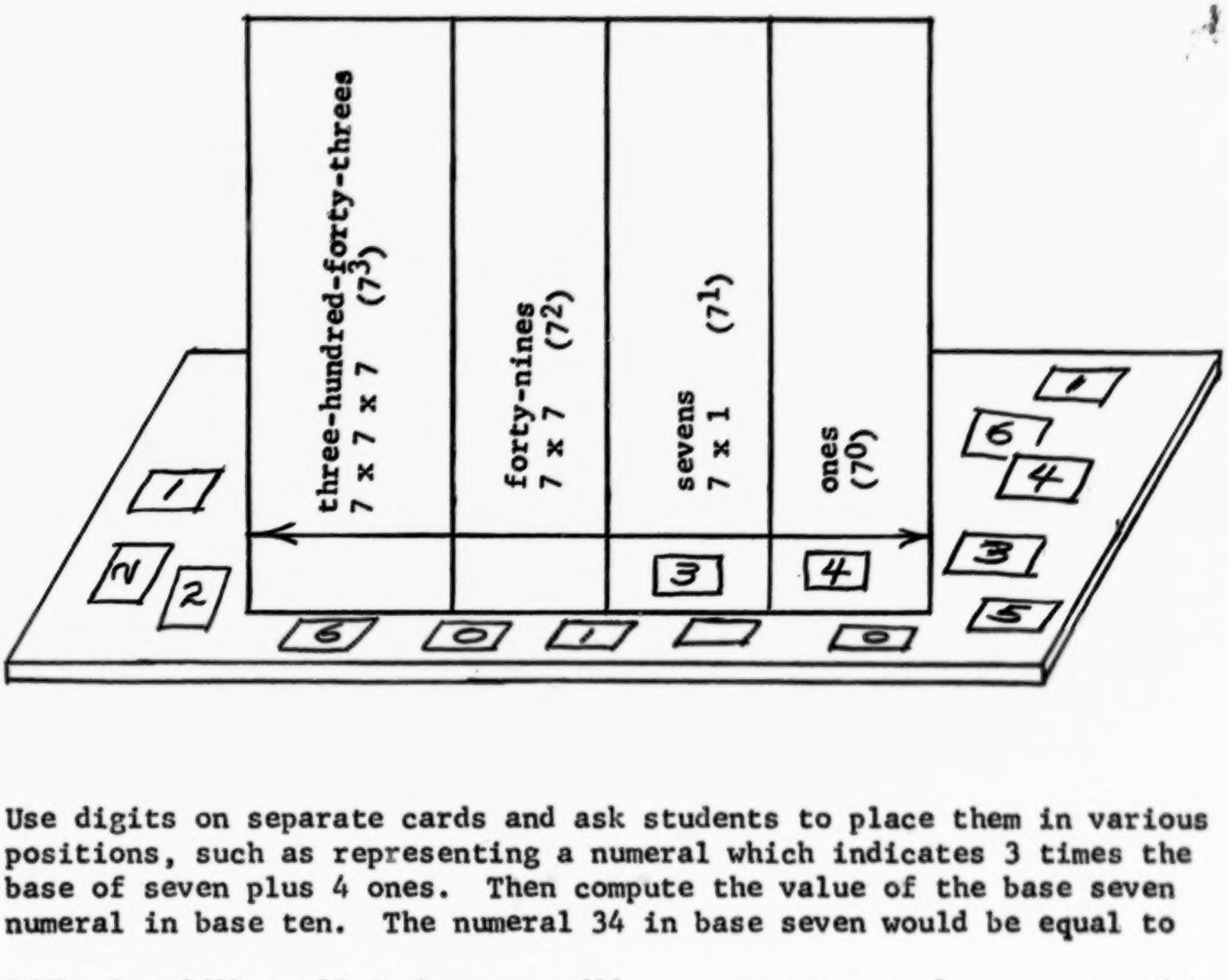
After 6 we have no more symbols to express additional quantities. Push down 7 ones, exchange for 1 seven bead. Write 10 = 1(7) + 0(1).



Repeat: push up 7 more ones. Exchange for 1 seven bead. Write 20 = 2(7) + 0(1).

Continue as far as the interests and abilities of the students warrant. It would be well to symbolize all the counting numbers as the beads are pushed up and regrouped. This demonstration should be done slowly and carefully so that the students understand the use of the model group in the numeral. Compare the numerals in the two bases. Show that the digit in the second position indicates the number of times the base is taken. For example: The digit 2 in the numeral 23 indicates 2 times the base. It does not matter what the base is.

The same emphasis should be put on the use of the base in the numeral when using place value charts, which can be made in any base. The illustration below shows a chart in base seven.



Use digits on separate cards and ask students to place them in various positions, such as representing a numeral which indicates 3 times the base of seven plus 4 ones. Then compute the value of the base seven numeral in base ten. The numeral 34 in base seven would be equal to

3(7) plus 4(1) or 25 in base ten (34_{seven} represents the same quantity as 25_{ten}).

READING NUMERALS IN BASE SEVEN

In order to read as well as to write numerals in another base we need to establish a method of naming them that is simple and meaningful. When reading numerals in another base, name the digit in the numeral. Do not call 13 in base seven "thirteen," because this is very confusing. Call the numeral 13 base seven "one three." Students should count aloud in base seven until they are familiar with the numerals: "one, two, three, four, five, six, one zero, one one, one two, one three, one four, one five, one six, two zero, two one, two two..."

The chart below compares large numerals in base seven to those in base ten. It should be noted that each position would be represented by the same numeral in both bases. Recall that the base seven positions are named according to their base ten value. Therefore, to save space the numerals are used rather than the word in base seven (49's names forty-nines' position).

10^6	10^5	10^4	10^3	10^2	10^1	10^0
1,000,000's	100,000's	10,000's	1,000's	100's	10's	1's
7^6	7^5	7^4	7^3	7^2	7^1	7^0
117,649's	16,807's	2,401's	343's	49's	7's	1's
	3	2	6	0	2	2

Above we have the numeral 326,022. Read it "three, two, six, zero, two, two." No matter what base it is written in, each digit has a value for the place in which it stands. The places for both base ten and base seven are indicated in the chart. For instance, hundreds' place in a base of ten is equal to only forty-nines' place in base seven. This is reasonable when we consider that the base is smaller. We can show the value in base ten with expanded notation. 326,022 = 3(100,000) + 2(10,000) + 6(1,000) + 0(100) + 2(10) + 2(1). In base ten we read this numeral as "three hundred twenty-six thousand, twenty-two." As indicated before, we do not have positional names for base seven. That is, we do not have a name that means to us "three-hundred-forty-threes" (7 x 7 x 7) as we have the word "thousand" that means 10 x 10 x 10 in base ten.

Therefore, we will read this numeral by simply naming the digits in order, "three, two, six, zero, two, two." If it is not understood that the base is seven, name the base after naming the digits.

The value of the base seven numeral 326,022 can be computed in base ten by using the positional value indicated in the chart.

Base seven 326,022 = 3(16,807) + 2(2,401) + 6(343) ÷ 0(49) ÷ 2(7) + 2(1) = 57,297 in base ten.

Read across the chart below. The second column indicates the base seven numeral for the equivalent amount which is represented in the first column in base ten. For example: 42 in base ten is represented as 60 (read it "six zero") in base seven.

BASE TEN		BASE SEVEN	
1		1	
2		2	
3		3	
4		4	
5		5	
6		6	
7		10	1 seven + 0 ones
8		11	1 seven + 1 one
9		12	1 seven + 2 ones
10	1 ten + 0 ones	13	1 seven + 3 ones
....	....	....	....
14	1 ten + 4 ones	20	2 sevens + 0 ones
15	1 ten + 5 ones	21	2 sevens + 1 one
....	....	....	....
21	2 tens + 1 one	30	3 sevens + 0 ones
22	2 tens + 2 ones	31	3 sevens + 1 one
....	....	....	....
41	4 tens + 1 one	56	5 sevens + 6 ones
42	4 tens + 2 ones	60	6 sevens + 0 ones
43	4 tens + 3 ones	61	6 sevens + 1 one
....	....	....	....
48	4 tens + 8 ones	66	6 sevens + 6 ones
49	4 tens + 9 ones	100	1 forty-nine + 0 sevens + 0 ones
50	5 tens + 0 ones	101	1 forty-nine + 0 sevens + 1 one
....	....	....	....
101	1 hundred + 0 tens + 1 one	203	2 forty-nines + 0 sevens + 3 ones
342	3 hundreds + 4 tens + 2 ones	666	6 forty-nines + 6 sevens + 6 ones
343	3 hundreds + 4 tens + 3 ones	1000	1 three-hundred-forty-three + 0 forty-nines + 0 sevens + 0 ones
....	....	....	....
2400	2 thousands + 4 hundreds + 0 tens + 0 ones	6666	6 three-hundred-forty-threes + 6 forty-nines + 6 sevens + 6 ones
2401	2 thousands + 4 hundreds + 0 tens + 1 one	10000	1 two-thousand-four-hundred-one + 0 three-hundred-forty-threes + 0 forty-nines + 0 ones

Written in expanded notation, the base ten numeral 42 = 4(10) + 2(1). The same quantity is represented by the base seven numeral 60 = 6(7) + 0(1). The two numerals represent the same quantity.

In order to understand the operations, students should develop a completed chart like the one above by writing all the counting numbers in expanded form, comparing the numerals in the two bases. This should be done at least up to the base squared (7^2) ("forty-nines"). To avoid writing out long place value names, the numerals could be used. For example: 343's instead of "three-hundred-forty-threes."

PLACE VALUE IN OTHER BASES

BASE	B^6	B^5	B^4	B^3	B^2	B^1	B^0
10	1,000,000's	100,000's	10,000's	1,000's	100's	10's	1's
7	117,649's	16,807's	2,401's	343's	49's	7's	1's
2	64's	32's	16's	8's	4's	2's	1's
3	729's	243's	81's	27's	9's	3's	1's
5	15,625's	3,125's	625's	125's	25's	5's	1's

The chart above indicates the base ten value of the positions in bases seven, two, three, and five. As in the previous charts, numerals have been used instead of the number names. A chart like this may be used to convert any base ten numeral to another base by following the procedure shown below, using the appropriate place value. To convert the base ten numeral 469 to a base seven numeral:

```
 469
-343    1 three-hundred-forty-three removed
 126
- 98    2 forty-nines removed
  28
- 28    4 sevens removed
   0    0 ones removed
```

Think: Take the largest place value "out of" 469 that is possible. This is 343. There will be 1(343) in the new numeral. Take out all that is possible of the next place. This is 98. There will be 2(49) in the new numeral. Take out 4(7). There are no ones remaining. The new base seven numeral will be 1(343) + 2(49) + 4(7) + 0(1) which is 1240_7. Since there are no ones in this numeral, the empty place must be indicated with a zero. Otherwise there would be only three positions in the base seven numeral, making it 124 which would be 1(49) + 2(7) + 4(1), the equivalent of only 67 in base ten (49 + 14 + 4 = 67).

THE NUMBER FACTS IN BASE SEVEN

The tables given below are constructed to show addition and multiplication "facts" in base seven. It is useful to make a table by referring to the counting chart made previously, and then to compare the numerals with those in the table.

It is difficult to "think" in a new base; therefore, in constructing the table, to find the base seven multiplication fact for 6 x 6, think: 6 x 6 = 36 in base ten. In the counting chart, 36 in base ten is written 51 (read it "five one") in base seven. Write 51 in the space for the product of 6 x 6. Expanded form shows 5(7) + 1(1) = 36. Do this for all the spaces. Remember that since this is a base of seven there are only seven digits. The numeral that represents seven (which is written 7 in base ten) is now written 10 in base seven. At the beginning, think the answer to problems in base ten and convert to base seven by referring to a counting chart. After this is done for a while, thinking in base seven will become natural.

ADDITION TABLE IN BASE SEVEN

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

MULTIPLICATION TABLE IN BASE SEVEN

X	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

ADDITION IN BASE SEVEN

Unfortunately, it is often possible to compute with little or no understanding of the processes involved. Such understanding can be developed by doing simple problems in another numeration system, since it is almost impossible to come to a solution without knowing the structure of the system. The following problems are computed in both base ten and in base seven. Regrouping in the base is indicated.

	BASE TEN		BASE SEVEN	
1.	4	4(1)	4	4(1)
	+ 2	2(1)	+ 2	2(1)
	6	6(1)	6	6(1)

Since there is no digit with a value over seven, the problem is the same in both base ten and base seven.

2.	4	4(1)	4	4(1)
	+ 3	3(1)	+ 3	3(1)
	7	7(1)	10	1(7) + 0(1)

Seven ones are regrouped in the base seven problem and the sum is written as 1 times the base plus 0 times one, with the numeral 10.

3.	3	3(1)	3	3(1)
	+ 8	8(1)	+ 11	1(7)
	11	1(10) + 1(1)	14	1(7) + 4(1)

In the problem 3 + 8 = 11 in base ten, 10 ones are regrouped as 1 ten. The sum is written as 1 times the base plus 1 times one, with the numeral 11. In the base seven problem there is no regrouping necessary because eight is written 11 in base seven. The sum is 1 times the base of seven plus 4 ones, or the numeral 14.

4.	23	2(10) + 3(1)	32	3(7) + 2(1)
	+ 23	2(10) + 3(1)	+ 32	3(7) + 2(1)
	46	4(10) + 6(1)	64	6(7) + 4(1)

There is no regrouping necessary in either problem. Is 6(7) + 4(1) equal to 4(10) + 6(1) in base ten?

The two examples below demonstrate regrouping in addition in base ten as compared with the same problem in base seven.

BASE TEN		BASE SEVEN	
34	3(10) + 4(1)	46	4(7) + 6(1)
+25	2(10) + 5(1)	+34	3(7) + 4(1)
59	5(10) + 9(1)	113	1(49) + 1(7) + 3(1)

In order to do the computation in base seven, the base seven "facts" are needed. 6 + 4 ones equals 7 + 3 ones. Write 3 in the ones' place in the sum. Regroup 7 ones as 1 seven. Add 4 + 3 + 1 sevens, which equals 7 + 1 sevens. Regroup 7 sevens as forty-nine. Write 1 in sevens' place in the sum. Since 7 sevens have been regrouped as 1 forty-nine, write 1 in forty-nines' place. The answer is 113, 1(49) + 1(7) + 3(1). Compute the value of base seven 113 to see if it equals 59 in base ten.

BASE TEN		BASE SEVEN	
342	3(100) + 4(10) + 2(1)	666	6(49) + 6(7) + 6(1)
+ 66	6(10) + 6(1)	+123	1(49) + 2(7) + 3(1)
408	4(100) + 0(10) + 8(1)	1122	1(343) + 1(49) + 2(7) + 2(1)

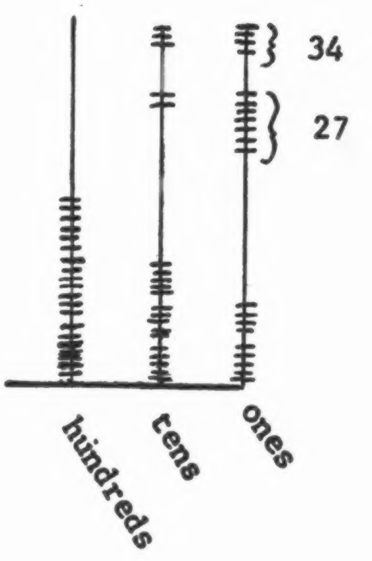
Sometimes it is helpful to think of the way each sum is written in another base when computing. Think: 6 + 3 = 9 in base ten; however, this is written as 12 in base seven. Write 2 in ones' place in the sum. Regroup ("carry") 1 group of seven. In base ten, 6 + 2 + 1 = 9. This is written as 12 in base seven. Again write 2 in the sum, in sevens' place. In base ten, 6 + 1 + 1 = 8. This is written in base seven as 11. Write 1 in forty-nines' place in the sum. Regroup seven forty-nines as 1 three-hundred-forty-three. The answer 1122 in base seven.

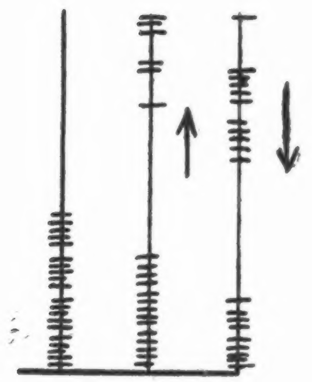
The twenty-bead abacus can be used to demonstrate the regrouping needed in different bases. The base ten example 34 + 27 is computed in both base ten and base seven below. The regrouping is demonstrated on the diagrams.

BASE TEN

34	3(10) + 4(1)
+27	2(10) + 7(1)
61	6(10) + 1(1)

BASE TEN ABACUS


34
27
hundreds
tens
ones

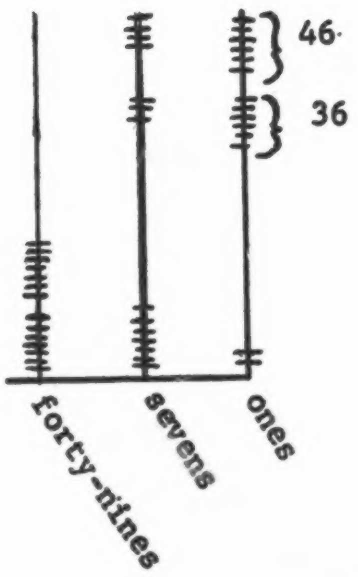


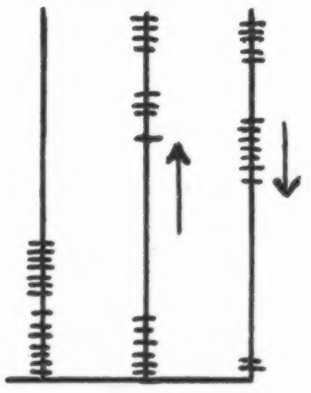
Exchange 10 ones for 1 ten.

BASE SEVEN

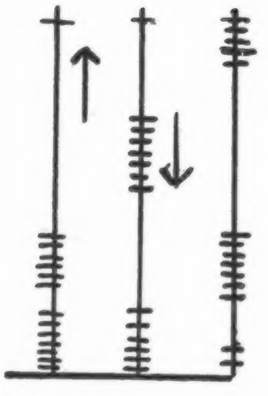
46	4(7) + 6(1)
+36	3(7) + 6(1)
115	1(49) + 1(7) + 5(1)

BASE SEVEN ABACUS


46
36
forty-nines
sevens
ones



Exchange 7 ones for 1 seven. There are now 7 sevens on sevens' column.



Exchange 7 sevens for 1 forty-nine.

ADDITION PROBLEMS IN BASE SEVEN

The following is a series of base seven addition problems of increasing difficulty. Such a series can be used either for enrichment or for understanding. Students should be encouraged to compare the computation in both bases and to understand the regrouping as it relates to the base of the system.

	BASE TEN		BASE SEVEN	
1.	$\begin{array}{r}5\\ \underline{5}\\ 10\end{array}$	1(10) + 0(1)	$\begin{array}{r}5\\ \underline{5}\\ 13\end{array}$	1(7) + 3(1)
2.	$\begin{array}{r}6\\ \underline{6}\\ 12\end{array}$	1(10) + 2(1)	$\begin{array}{r}6\\ \underline{6}\\ 15\end{array}$	1(7) + 5(1)
3.	$\begin{array}{r}7\\ \underline{7}\\ 14\end{array}$	1(10) + 4(1)	$\begin{array}{r}10\\ \underline{10}\\ 20\end{array}$	2(7) + 0(1)
4.	$\begin{array}{r}8\\ \underline{8}\\ 16\end{array}$	1(10) + 6(1)	$\begin{array}{r}11\\ \underline{11}\\ 22\end{array}$	2(7) + 2(1)
5.	$\begin{array}{r}9\\ \underline{9}\\ 18\end{array}$	1(10) + 8(1)	$\begin{array}{r}12\\ \underline{12}\\ 24\end{array}$	2(7) + 4(1)
6.	$\begin{array}{r}10\\ \underline{10}\\ 20\end{array}$	2(10) + 0(1)	$\begin{array}{r}13\\ \underline{13}\\ 26\end{array}$	2(7) + 6(1)
7.	$\begin{array}{r}11\\ \underline{11}\\ 22\end{array}$	2(10) + 2(1)	$\begin{array}{r}14\\ \underline{14}\\ 31\end{array}$	3(7) + 1(1)
8.	$\begin{array}{r}12\\ \underline{12}\\ 24\end{array}$	2(10) + 4(1)	$\begin{array}{r}15\\ \underline{15}\\ 33\end{array}$	3(7) + 3(1)
9.	$\begin{array}{r}13\\ \underline{13}\\ 26\end{array}$	2(10) + 6(1)	$\begin{array}{r}16\\ \underline{16}\\ 35\end{array}$	3(7) + 5(1)
10.	$\begin{array}{r}14\\ \underline{14}\\ 28\end{array}$	2(10) + 8(1)	$\begin{array}{r}20\\ \underline{20}\\ 40\end{array}$	4(7) + 0(1)
11.	$\begin{array}{r}35\\ \underline{35}\\ 70\end{array}$	7(10) + 0(1)	$\begin{array}{r}50\\ \underline{50}\\ 130\end{array}$	1(49) + 3(7) + 0(1)

The addition problems below are computed in both base ten and base seven. Unless the base seven notation is thoroughly mastered, each sum should be checked as indicated by comparing the sum with the equivalent base ten value.

BASE TEN		BASE SEVEN	
$\begin{array}{r}12\\ \underline{71}\\ 83\end{array}$	$8(10) + 3(1) = 83$	$\begin{array}{r}15\\ \underline{131}\\ 146\end{array}$	$1(49) + 4(7) + 6(1) = 83$

1.	$\begin{array}{r}39\\ \underline{39}\\ 78\end{array}$	$\begin{array}{r}54\\ \underline{54}\\ 141\end{array}$	11.	$\begin{array}{r}44\\ \underline{34}\\ 78\end{array}$	$\begin{array}{r}62\\ \underline{46}\\ 141\end{array}$
2.	$\begin{array}{r}18\\ \underline{18}\\ 36\end{array}$	$\begin{array}{r}24\\ \underline{24}\\ 51\end{array}$	12.	$\begin{array}{r}52\\ \underline{16}\\ 68\end{array}$	$\begin{array}{r}103\\ \underline{22}\\ 125\end{array}$
3.	$\begin{array}{r}23\\ \underline{41}\\ 64\end{array}$	$\begin{array}{r}32\\ \underline{56}\\ 121\end{array}$	13.	$\begin{array}{r}49\\ \underline{23}\\ 72\end{array}$	$\begin{array}{r}100\\ \underline{32}\\ 132\end{array}$
4.	$\begin{array}{r}32\\ \underline{15}\\ 47\end{array}$	$\begin{array}{r}44\\ \underline{21}\\ 65\end{array}$	14.	$\begin{array}{r}61\\ \underline{48}\\ 109\end{array}$	$\begin{array}{r}115\\ \underline{66}\\ 214\end{array}$
5.	$\begin{array}{r}24\\ \underline{12}\\ 36\end{array}$	$\begin{array}{r}33\\ \underline{15}\\ 51\end{array}$	15.	$\begin{array}{r}38\\ \underline{52}\\ 90\end{array}$	$\begin{array}{r}53\\ \underline{103}\\ 156\end{array}$
6.	$\begin{array}{r}16\\ \underline{13}\\ 29\end{array}$	$\begin{array}{r}22\\ \underline{16}\\ 41\end{array}$	16.	$\begin{array}{r}87\\ \underline{98}\\ 185\end{array}$	$\begin{array}{r}153\\ \underline{200}\\ 353\end{array}$
7.	$\begin{array}{r}19\\ \underline{12}\\ 31\end{array}$	$\begin{array}{r}25\\ \underline{15}\\ 43\end{array}$	17.	$\begin{array}{r}98\\ \underline{98}\\ 196\end{array}$	$\begin{array}{r}200\\ \underline{200}\\ 400\end{array}$
8.	$\begin{array}{r}13\\ \underline{21}\\ 34\end{array}$	$\begin{array}{r}16\\ \underline{30}\\ 46\end{array}$	18.	$\begin{array}{r}57\\ \underline{23}\\ 80\end{array}$	$\begin{array}{r}111\\ \underline{32}\\ 143\end{array}$
9.	$\begin{array}{r}19\\ \underline{23}\\ 42\end{array}$	$\begin{array}{r}25\\ \underline{32}\\ 60\end{array}$	19.	$\begin{array}{r}61\\ \underline{16}\\ 77\end{array}$	$\begin{array}{r}115\\ \underline{22}\\ 140\end{array}$
10.	$\begin{array}{r}33\\ \underline{20}\\ 53\end{array}$	$\begin{array}{r}45\\ \underline{26}\\ 104\end{array}$	20.	$\begin{array}{r}29\\ \underline{36}\\ 65\end{array}$	$\begin{array}{r}41\\ \underline{51}\\ 122\end{array}$

MULTIPLICATION IN BASE SEVEN

The examples below are computed in both base ten and base seven. Notice that the familiar algorism is the same in both bases; however, the base seven notation must be used throughout when computing in that base.

	BASE TEN	BASE SEVEN
1.	$\begin{array}{rl} 8 & \\ \times 3 & \\ \cline{1-1} 24 & 2(10) + 4(1) \end{array}$	$\begin{array}{rl} 11 & \\ \times 3 & \\ \cline{1-1} 33 & 3(7) + 3(1) \end{array}$
	Think: 3 x 8 ones equals 24 ones which are regrouped as 2(10) + 4(1).	Think: 3 x 1 one equals 3 ones, and 3 x 1 seven equals 3 sevens.
2.	$\begin{array}{rl} 15 & \\ \times 4 & \\ \cline{1-1} 60 & 6(10) + 0(1) \end{array}$	$\begin{array}{rl} 21 & \\ \times 4 & \\ \cline{1-1} 114 & 1(49) + 1(7) + 4(1) \end{array}$
	Think: 4 x 5 ones equals 20 ones which are regrouped as 2 tens. 4 x 1 ten equals 4 tens. 4 tens plus the 2 tens that are regrouped equals 6 tens.	Think: 4 x 1 one equals 4 ones. 4 x 2 sevens equals 11 (eight) sevens which are regrouped as 1 forty-nine plus 1 seven.
3.	$\begin{array}{rl} 53 & \\ \times 24 & \\ \cline{1-1} 212 & 4 \times 53 \\ 106 & 20 \times 53 \\ \cline{1-1} 1272 & 1(1000) + 2(100) + \\ & 7(10) + 2(1) \end{array}$	$\begin{array}{rl} 104 & \\ \times 33 & \\ \cline{1-1} 315 & 3 \times 104 \\ 315 & 30 \times 104 \\ \cline{1-1} 3465 & 3(343) + 4(49) + \\ & 6(7) + 5(1) \end{array}$

(Partial products are used in base seven in the same manner as in base ten.)

	BASE TEN	BASE SEVEN
4.	$\begin{array}{rl} 79 & \\ \times 7 & \\ \cline{1-1} 553 & 5(100) + 5(10) + \\ & 3(1) \end{array}$	$\begin{array}{rl} 142 & \\ \times 10 & \\ \cline{1-1} 1420 & 1(343) + 4(49) + 2(7) + \\ & 0(1) \end{array}$

(Notice that the "short way" of multiplying by 10 is the same in base seven as in base ten. Remember that the numeral 10 represents one of the base of the system in both base seven and base ten.)

The following problems might be computed in base ten, then converted to base seven and computed in that base. The products can be checked by comparison to base ten, as indicated.

	BASE TEN	BASE SEVEN	
1.	$\begin{array}{r} 49 \\ 3 \\ \hline 147 \end{array}$	$\begin{array}{r} 100 \\ 3 \\ \hline 300 \end{array}$	3(49) + 0(7) + 0(1) = 147
2.	$\begin{array}{r} 24 \\ 2 \\ \hline 48 \end{array}$	$\begin{array}{r} 33 \\ 2 \\ \hline 66 \end{array}$	6(7) + 6(1) = 48
3.	$\begin{array}{r} 14 \\ 4 \\ \hline 56 \end{array}$	$\begin{array}{r} 20 \\ 4 \\ \hline 110 \end{array}$	1(49) + 1(7) + 0(1) = 56
4.	$\begin{array}{r} 21 \\ 6 \\ \hline 126 \end{array}$	$\begin{array}{r} 30 \\ 6 \\ \hline 240 \end{array}$	2(49) + 4(7) + 0(1) = 126
5.	$\begin{array}{r} 35 \\ 7 \\ \hline 245 \end{array}$	$\begin{array}{r} 50 \\ 10 \\ \hline 500 \end{array}$	5(49) + 0(7) + 0(1) = 245
6.	$\begin{array}{r} 42 \\ 8 \\ \hline 336 \end{array}$	$\begin{array}{r} 60 \\ 11 \\ \hline 60 \\ 60 \\ \hline 660 \end{array}$	6(49) + 6(7) + 0(1) = 336
7.	$\begin{array}{r} 349 \\ 8 \\ \hline 2792 \end{array}$	$\begin{array}{r} 1006 \\ 11 \\ \hline 1006 \\ 1006 \\ \hline 11066 \end{array}$	1(2401) + 1(343) + 0(49) + 6(7) + 6(1) = 2792
8.	$\begin{array}{r} 1072 \\ 12 \\ \hline 2144 \\ 1072 \\ \hline 12864 \end{array}$	$\begin{array}{r} 3061 \\ 15 \\ \hline 21425 \\ 3061 \\ \hline 52335 \end{array}$	5(2401) + 2(343) + 3(49) + 3(7) + 5(1) = 12864
9.	$\begin{array}{r} 364 \\ 14 \\ \hline 1456 \\ 364 \\ \hline 5096 \end{array}$	$\begin{array}{r} 1030 \\ 20 \\ \hline 20600 \end{array}$	2(2401) + 0(343) + 6(49) + 0(7) + 0(1) = 5096
10.	$\begin{array}{r} 98 \\ 4 \\ \hline 392 \end{array}$	$\begin{array}{r} 200 \\ 4 \\ \hline 1100 \end{array}$	1(343) + 1(49) + 0(7) + 0(1) =392

SUBTRACTION IN BASE SEVEN

The algorisms for subtraction in base seven are the same as in base ten, except that the regrouping is done with a model group of seven instead of ten. The familiar "borrowing" algorism is shown below.

$$\begin{array}{r} 22 \\ -\ 6 \\ \hline 16 \end{array} \qquad\qquad \begin{array}{lr} & 1\text{ ten}\quad 2 + 10\text{ ones} \\ & \cancel{2}\text{ tens} + \cancel{2}\text{ ones} \\ - & 6\text{ ones} \\ \hline & 1\text{ ten} + 6\text{ ones} \end{array}$$

It may increase understanding if the 1 ten which is "borrowed" (regrouped as 10 ones) is thought of as being added to the ones which are already in ones' place. Writing the numeral 1 beside the 2 makes the regrouped sum of 12 because 1(10) + 2(1) = 12. This should be thoroughly understood.

BASE TEN | BASE SEVEN

$$\begin{array}{r} \\ 22 \\ -\ 6 \\ \hline 16 \end{array} \qquad\qquad \begin{array}{r} \\ 31 \\ -\ 6 \\ \hline 22 \end{array} \quad \begin{array}{l} 2\text{ sevens}\quad 1 + 7\text{ ones (or 8 ones)} \\ \cancel{3}\text{ sevens} + \cancel{1}\text{ ones} \\ \qquad\qquad\quad 6\text{ ones} \\ \hline 2\text{ sevens} + 2\text{ ones} = 16 \end{array}$$

The same concept is presented in the base seven problem above. When 1 seven is "borrowed" or regrouped as 7 ones, the value of ones' place is the sum of 1 seven + 1 one or eight, which is written as 11 in base seven. The subtraction fact 11 - 6 = 2 is needed to compute the difference in ones' place.

An understanding of subtraction can be gained by computing in other bases. Other methods can be developed, such as equal additions and complements, as experimental and enrichment activities.

LONG DIVISION IN BASE SEVEN

It is possible to do division in base seven either as multiple subtraction or in the form $\overline{)\qquad\qquad}$, sometimes called the "divided by" form. Multiple subtraction is easier for beginners, as it follows from the concepts presented under subtraction. The same examples are computed in both bases below.

	BASE TEN	BASE SEVEN			
1.	7 r. 3	10 r. 3			
	7/52	10/103			1 forty-nine + 0 sevens + 3 ones
	49	-10	1		- 1 seven + 0 ones
	3	63			6 sevens + 3 ones
		-10	1		- 1 seven + 0 ones
		53			5 sevens + 3 ones
		-10	1		- 1 seven + 0 ones
		43			4 sevens + 3 ones
		-10	1		- 1 seven + 0 ones
		33			3 sevens + 3 ones
		-10	1		- 1 seven + 0 ones
		23			2 sevens + 3 ones
		-10	1		- 1 seven + 0 ones
		13			1 seven + 3 ones
		-10	1		- 1 seven + 0 ones
		3	10	1(7)+0(1)	3 ones
		10		1(7)+0(1)	
	x7	x10		1(7)+0(1)	
	49	100		1(49)+0(7)+0(1)	
	+3	+ 3			
	52	103		1(49)+0(7)+3(1) = 52 in base ten	
2.	12 r.2	15 r.2			
	8/98	11/200			
		-66	6		
		101			
		-66	6		
		2	15	1(7) + 5(1) = 12	
	12	15			
	x8	x11			
	96	15			
	+2	15			
	98	165			
		+ 2			
		200			

Think: 1 x 5 = 5, 1 x 1 = 1, and repeat. Add the remainder. Remember, 5 + 2 = 10, or one group of seven. "Carry" 1 seven. This makes 7 sevens, including the one you are "carrying." Put down the zero and "carry" again. You now have 1 + 1 = 2. Your answer is 2(49) + 0(7) + 0(1).

DIVISION PROBLEMS IN BASE SEVEN

Remember, as you work in base seven, you must either convert each number to base seven or else think in the base.

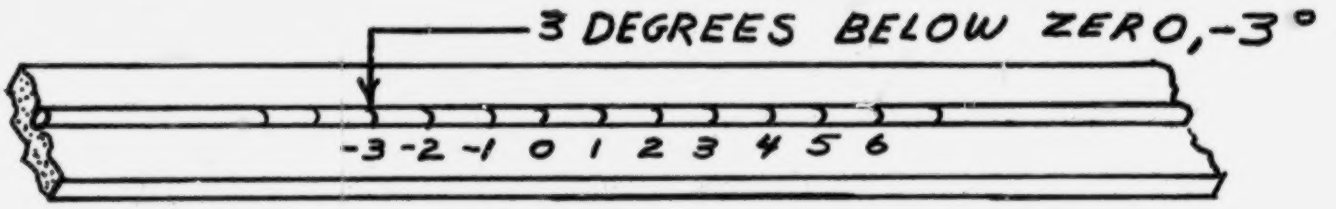
BASE TEN	BASE SEVEN
1. 11 r. 3	14 r. 3
9/102	12/204
-63 7	-120 10
39	54
-36 +4	- 51 + 4
3 11	3 14 1(7) + 4(1) = 11
Check: 11	14
x 9	x12
99	31
+ 3	14
102	201
	+ 3
	204 2(49) + 0(7) + 4(1) = 102
2. 29 r. 3	41 r. 3
7/206	10/413
-49 7	-100 10
157	313
-49 7	-100 10
108	213
-49 7	-100 10
59	113
-49 7	-100 10
10	13
- 7	-10 1
3 1	3 41 4(7) + 1(1) = 29
Check: 29	41
x 7	x10
203	410
+ 3	+ 3
206	413 4(49) + 1(7) + 3(1) = 206
3. 40 r. 10	55 r. 13
12/490	15/1300
48	114
10	130
	114
	13
Check: 40	55
x12	x15
80	404
40	55
480	1254
10	13
490	1300 1(343) + 3(49) + 0(7) + 0(1) = 490

PART IV. TEACHING THE OPERATIONS WITH LEARNING AIDS

NUMBER SYSTEMS

At the elementary levels we are concerned with the four fundamental operations of addition, subtraction, multiplication, and division. For the most part, elementary mathematics confines the operations within the system of natural numbers (positive integers), zero, and positive rational (fractional) numbers. Modern instructional programs incorporate the use of negative numbers and so include operations with positive and negative integers, zero, and positive and negative rational numbers.

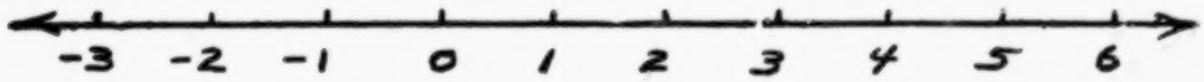
As has been pointed out, new number systems have been contrived as man has found a need for them. Negative numbers are needed to describe physical situations such as "below zero" temperatures and to furnish an answer to expressions of the type 2 - 3 = ? A horizontal thermometer can be used as a number line to illustrate both situations.



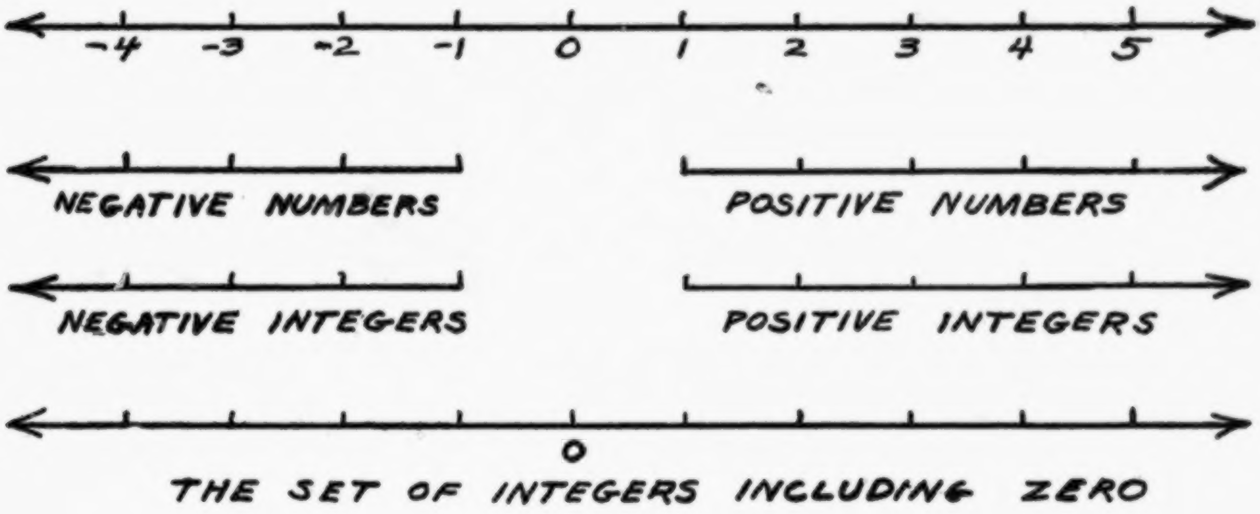
A measure of the temperature three degrees below zero would be represented by the point corresponding to -3° on the "thermometer number line." Elementary children too often have been told that "there is no answer" to such expressions as 2 - 3 = ? or that one "always has to put the larger number first in a subtraction problem." It can be seen on the number line that if the temperature dropped 3° from 2° above zero, the reading would be -1°. The example 2 - 3 = -1 could be employed to represent this situation.

A number line is one of the most versatile and useful means of representing the real number system and should be so constructed that it presents a true picture (graph) of the numbers which are used in computation. It should be considered as a geometric line consisting of an infinite set of points and continuing without interruption in both directions.

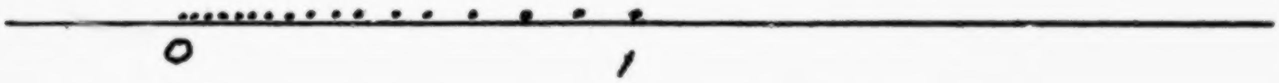
A point, zero, is chosen at random on the line. One is represented as an arbitrary distance to the right of zero (0 —— 1). Segments of the length from 0 to 1 are then used to establish the successive points of the line that represent the counting numbers, often called the natural numbers. It should be remembered that one and only one point will correspond to each number. As the number line is to represent the negative numbers also, the line is extended to the left of zero in successive unit lengths to establish points which represent the negative integers.


-3
-2
-1
0
1
2
3
4
5
6

The line now represents the integers, the positive and negative numbers, and zero, which is considered neither positive or negative.


-4
-3
-2
-1
0
1
2
3
4
5
NEGATIVE NUMBERS
POSITIVE NUMBERS
NEGATIVE INTEGERS
POSITIVE INTEGERS
0
THE SET OF INTEGERS INCLUDING ZERO

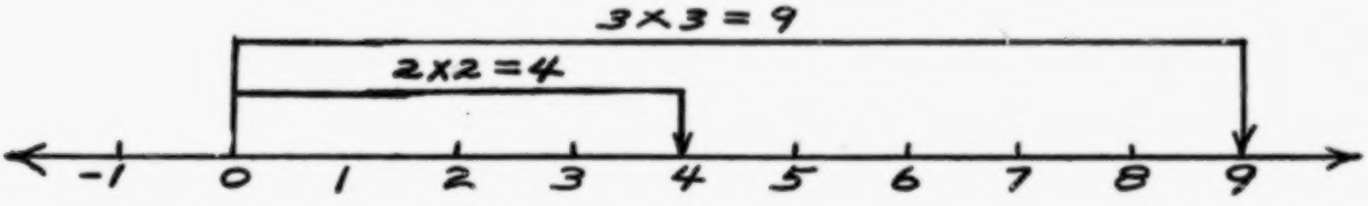
It is evident that it would be possible to locate points representing any fractional part of a unit length from 0 to 1 by locating points corresponding to successively smaller divisions.



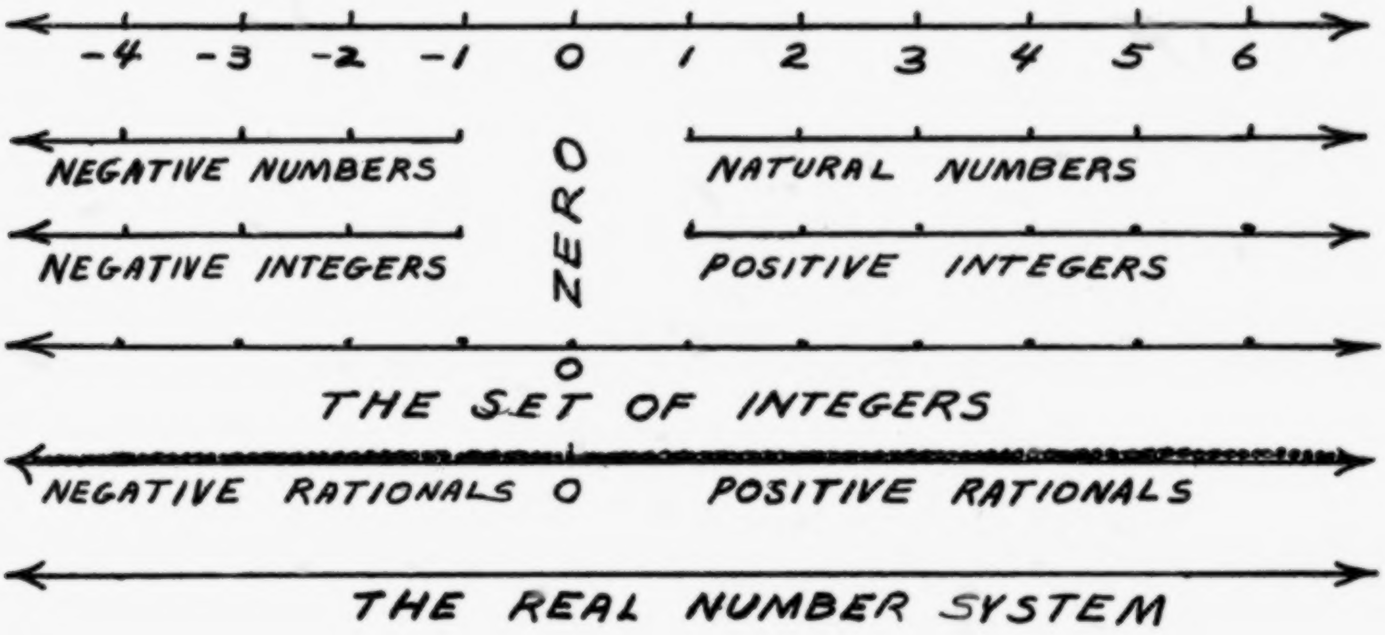
These are the rational numbers, it being understood that a rational number can be expressed as the quotient of two integers (2/1, 1/3, 4/5, -2/7...), it being understood that division by zero is undefined, of the form a/b ($b \neq 0$). Number lines can be made to represent all the numbers commonly used for computation in elementary mathematics.

Since the number line is a continuous geometric line made up of an infinite set of points, it may also be seen to represent the irrational numbers, those numbers which cannot be expressed as the quotient of two integers.

Consider the expression $\sqrt{4}$ (read "the square root of 4"), indicating that the number is to be found which multiplied by itself equals 4. This number is 2. It is found on the number line at the point 2. However, $\sqrt{5}$ could not be identified as an exact point of a number line consisting of only points corresponding to the rational number that multiplied by itself will give us 5.


3×3 = 9
2×2 = 4
-1
0
1
2
3
4
5
6
7
8
9

In this way the line represents the set of positive integers, zero, the negative integers, the rationals and the irrationals. These combine to represent the real number system which can be shown by a set of points, as in the following:


-4
-3
-2
-1
0
1
2
3
4
5
6
NEGATIVE NUMBERS
NATURAL NUMBERS
ZERO
NEGATIVE INTEGERS
POSITIVE INTEGERS
0
THE SET OF INTEGERS
NEGATIVE RATIONALS
0
POSITIVE RATIONALS
THE REAL NUMBER SYSTEM

It is not intended that elementary students be totally familiar with all these subsets of the number system, but it is desirable that students be aware that such subsets exist. Even primary children can see the advantage of writing -3° instead of "three degrees below zero."

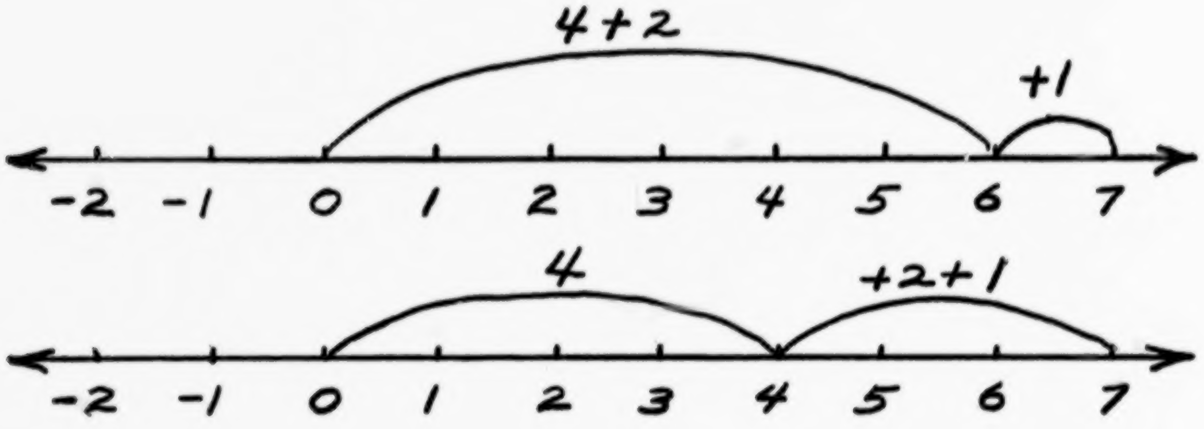
The four fundamental operations are not haphazard or accidental and would be impossible if it were not for the laws that lay down the rules by which these operations are carried out. Students should have an acquaintance with these laws not only to develop basic mathematical concepts but also to aid in the understanding and mastery of computational skills. We will be considering the associative, commutative, and distributive laws or principles, also closure, and the identity properties of our number system as we deal with the development of the operations.

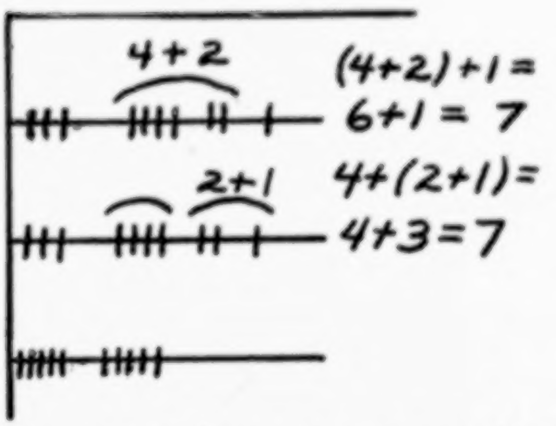
THE OPERATION OF ADDITION

Exercises in the operation of addition are usually presented as a memory or drill lesson, with little or no attention given to the structural properties which make it possible. Learning aids can do much to develop the basic concepts needed to give students a better understanding and to increase their retention of the "facts."

Addition is a binary operation--an operation on two numbers which produces a third number, called the sum. To add $3 + 2 + 1$ we find the sum of $3 + 2$, which is 5, and then add $5 + 1 = 6$. The sum could also be computed as $3 + 2 + 1$ by combining $2 + 1$ as 3 then adding $3 + 3$, yielding the sum 6. In either case the sum will be the same. The grouping of 3, 2, and 1 in addition may be written as $(3 + 2) + 1 = 6$, or $3 + (2 + 1) = 6$. The associative law states that, without changing the order of the addends, any grouping is possible to find the sum.

The example $4 + 2 + 1 = 7$ may be added as $(4 + 2) + 1 = 6 + 1 = 7$ or as $4 + (2 + 1) = 4 + 3 = 7$. It does not make any difference with which "neighbor" the addends are paired. This may be demonstrated by the number line, the counting frame, counting sticks, and other learning aids.

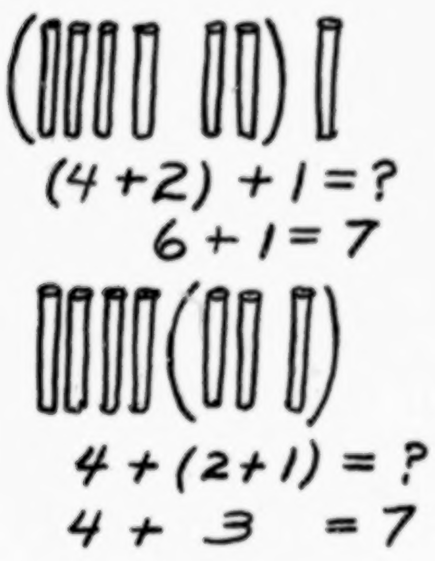


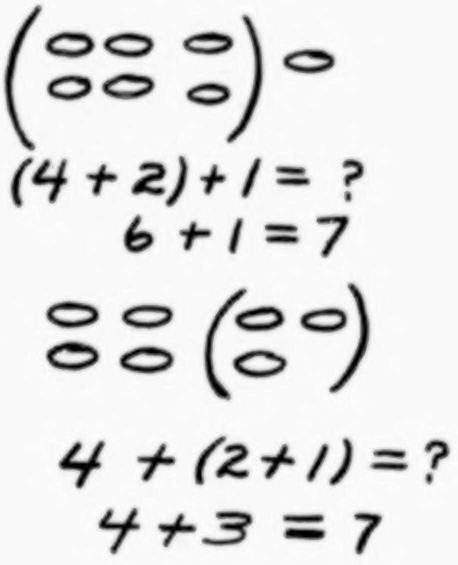


Emphasize that 4 + 2 names 6. then add 6 + 1 and name the sum 7.

Emphasize that 2 + 1 names 3. then add 4 + 3 and name the sum 7.

Counting devices are excellent to demonstrate different names for the same quantity, as shown below(4 + 2 for 6, 2 + 1 for 3...).


(4 + 2) + 1 = ?
6 + 1 = 7
4 + (2 + 1) = ?
4 + 3 = 7


(4 + 2) + 1 = ?
6 + 1 = 7
4 + (2 + 1) = ?
4 + 3 = 7

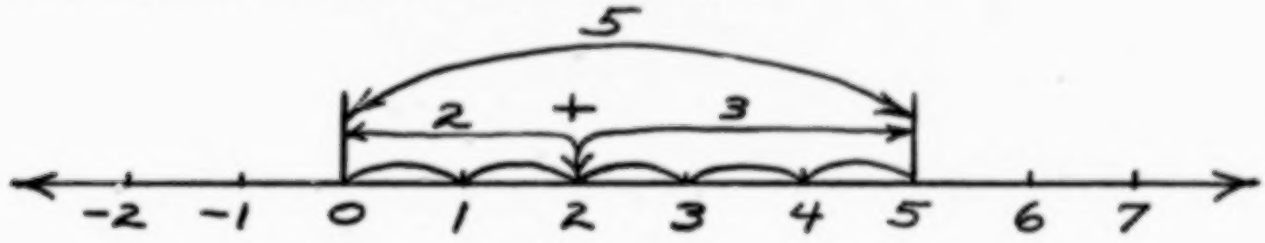
The use of the associative property can be demonstrated by the regrouping in the following examples.

$$\begin{aligned} 18 + 3 &= (10 + 8) + 3 \\ &= 10 + (8 + 3) \\ &= 10 + 11 \\ &= 21 \end{aligned} \qquad \begin{array}{r} 18 \\ \underline{+\ 3} \\ 11 \\ \underline{10} \\ 21 \end{array} \quad (8 + 3) \qquad \begin{array}{r} 1 \\ 18 \\ \underline{+\ 3} \\ 21 \end{array}$$

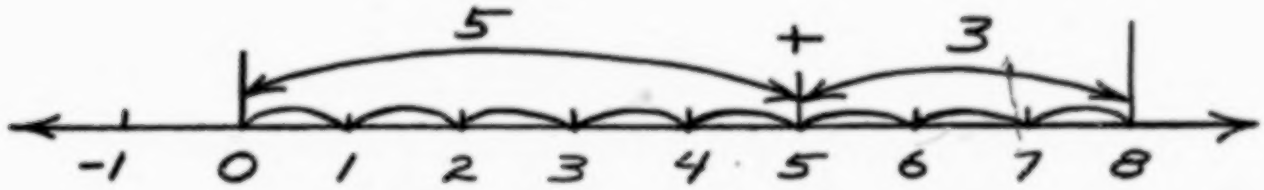
1 (10 ones regrouped as 1 ten)

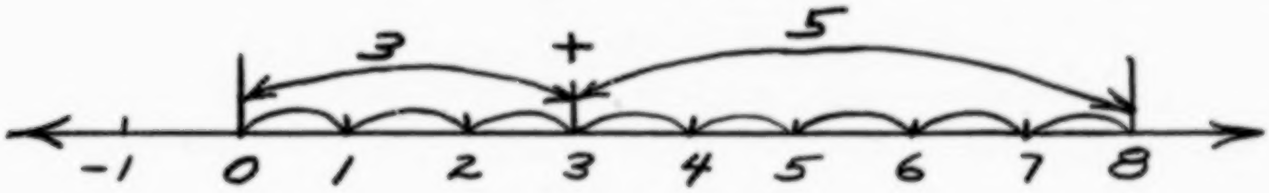
If it were not for the property of closure in the number system, it would be impossible to express all sums which are needed. The word closure is used here in the sense of "being included in." The property of closure indicates that an operation performed within the system will yield a number which is also in the system. The operation of addition is closed within the set of natural numbers. This means that the sum of any two natural numbers will always be a natural number. We may illustrate this with the number line. When the line is considered as a graph of the real number system extending without interruption in both directions, it can be seen that the sum of any number of addends may be expressed with a third number and this number will be in the system of natural numbers. Considering that addition is a binary operation, we assume that: For any pair of natural numbers there exists a natural number which expresses their sum.

The property of closure also assumes that the sum of any pair of natural numbers is a unique number. On the number line, 2 + 3 = 5 may be thought of as a process of counting.



The commutative law states that the order of the addends makes no difference in the sum. It can be demonstrated on the number line that 5 + 3 = 8 and that 3 + 5 = 8.

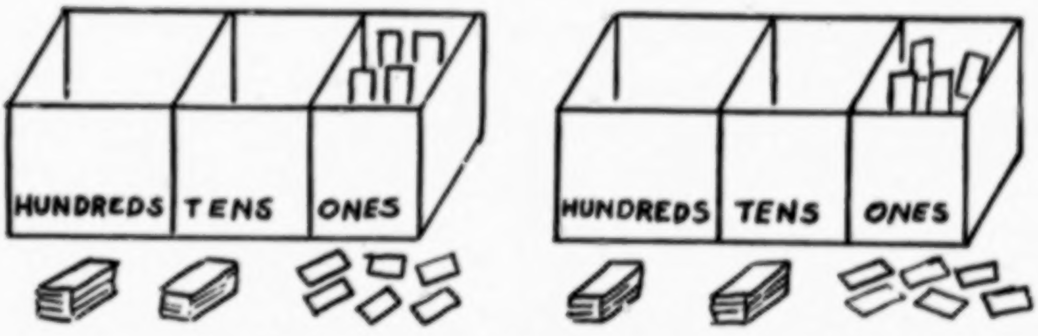




+	0	1	2	3	4	5	6	7	8	9
0	0									
1	1	2								
2	2	3	4							
3	3	4	5	6						
4	4	5	6	7	8				12	
5	5	6	7	8	9	10				
6	6	7	8	9	10	11	12			
7	7	8	9	10	11	12	13	14		
8	8	9	10	11	12	13	14	15	16	
9	9	10	11	12	13	14	15	16	17	18

An application of the commutative law reduces the number of addition facts necessary to commit to memory. Completion of the chart above would result in an image of the facts which are shown. For example, the intersection of the "8" column and the "4" row yields the same sum as the intersection of the "4" column and the "8" row.

The number 0 has a special property in that the addition of zero to any natural number does not change the number. Zero is called the identity element (number) for the operation of addition. If an empty set is combined with a set of 4, or a set of 4 is combined with an empty set (addition is commutative), the sum is 4. Many concrete devices can be used to demonstrate the identity element 0. In the illustration below there are 4 ones in the ones' pocket. As previously stated, 0 represents the absence of quantity. If 0 ones are added, the ones' pocket still contains 4 ones. Similar demonstrations can be done with counting sticks, flannelboard cutouts, and so forth.



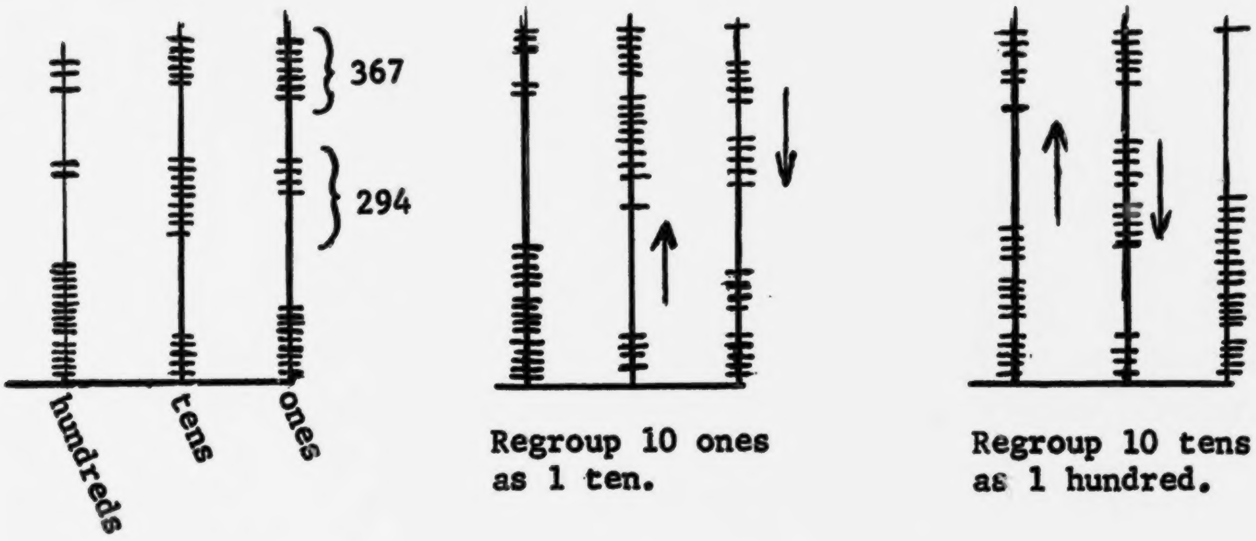
4 ONES PLUS 0 ONES = 4 ONES
4(ONE) + 0(ONE) = 4

Addition is combining groups by irregular counting and regrouping in the base of the numeration system. In base ten this is done in groups of ten and powers of ten. Students should understand that they are manipulating groups rather than simply memorizing facts. Instead of only remembering the sum of 8 and 5, a student should understand that a group of 8 and a group of 5 can be regrouped as a group of 1 ten and a group of 3 ones, and that this regrouping is represented by the numeral 13. Two algorisms are suggested below which show ones and tens regrouped in addition.

↙——10 tens regrouped as 1 hundred
11←—10 ones regrouped as 1 ten
367
294
661

367
294
11 (7 + 4 = 11)
150 (60 + 90 = 150)
500 (300 + 200 = 500)
661

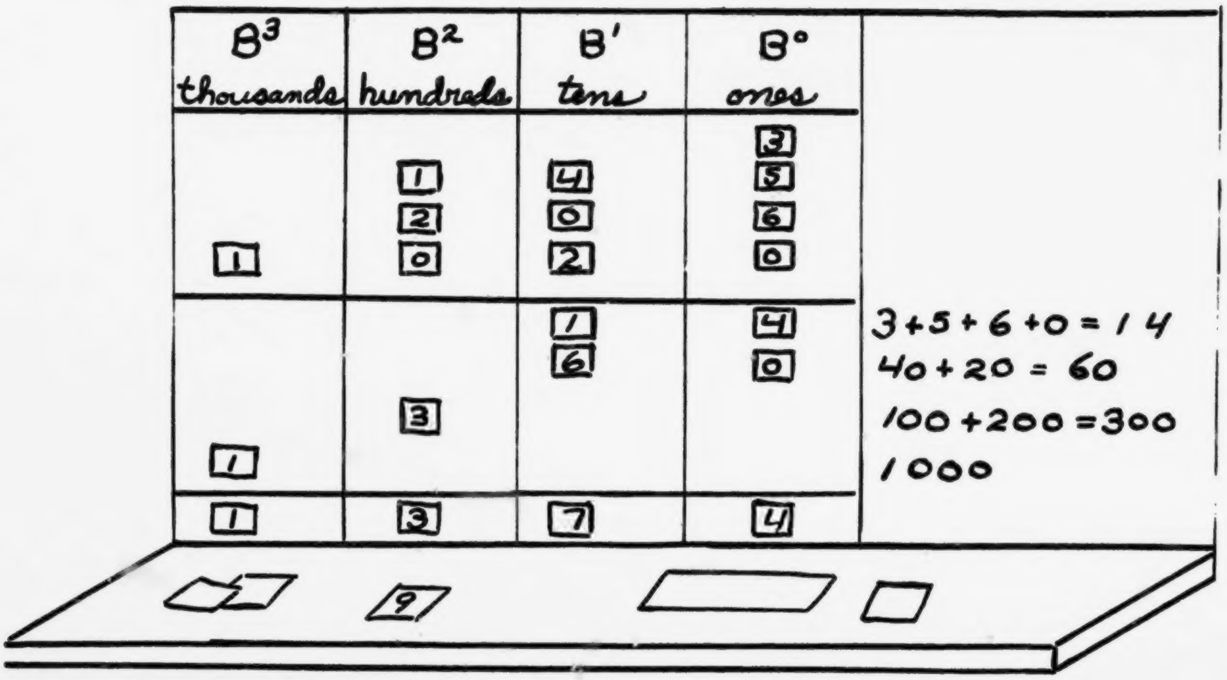
The regrouping for this example can be demonstrated on an abacus as shown in the illustration below:



Regroup 10 ones as 1 ten.

Regroup 10 tens as 1 hundred.

Place value charts on chalkboard panels are useful to help students understand regrouping and positional notation in "ragged" addition, such as 3 + 145 + 206 + 1020.



The chart can be constructed to any degree of sophistication desired. A place value chart employing magnetized numerals which can be arranged to indicate grouping and regrouping is particularly useful. It should, however, not be used as a "gadget" for mechanically representing the algorism, but to develop real understanding of the reason for "keeping the columns straight." Students should learn that the digits 2 and 0 are in the column represented by the base squared (10^2), not simply that this is hundreds' place, and so on. The regrouping can be illustrated on the chalkboard as indicated.

Magnetic or flannelboard numerals may be used to make displays similar to the arrangements below. The number pairs which yield the 100 "facts" when the operation of addition is performed on the natural numbers and zero are shown in Table A. It is 10 pairs wide and 10 pairs long. In Table B, the sums yielded under the operation have been substituted for each of the number pairs in Table A.

Many patterns may be investigated. For example, the broken diagonal represents the pairs that form the possible combinations which yield the sum 9. The solid diagonal transverses the "doubles." With manipulative numerals children may remove and display such things as: All the number pairs that yield the sum of 8; the sums that are doubles; all the addition facts that are needed to compute 23 + 85; and so on.

+	0	1	2	3	4	5	6	7	8	9
0	00	01	02	03	04	05	06	07	08	09
1	10	11	12	13	14	15	16	17	18	19
2	20	21	22	23	24	25	26	27	28	29
3	30	31	32	33	34	35	36	37	38	39
4	40	41	42	43	44	45	46	47	48	49
5	50	51	52	53	54	55	56	57	58	59
6	60	61	62	63	64	65	66	67	68	69
7	70	71	72	73	74	75	76	77	78	79
8	80	81	82	83	84	85	86	87	88	89
9	90	91	92	93	94	95	96	97	98	99

TABLE A

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

TABLE B

THE OPERATION OF SUBTRACTION

Subtraction is the inverse operation of addition. Subtraction "undoes" addition. Although we are accustomed to memorizing the subtraction facts, this is not necessary when addition and subtraction are considered as inverse operations of each other. A portion of the chart will make this evident.

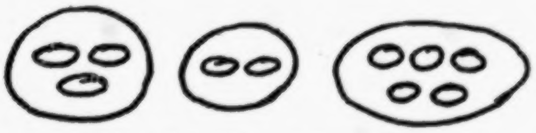
inverse operation →

$\overset{-}{+}$	0	1	2	3	4
0	0				
1	1	2			
2	2	3	4		
3	3	4	$\boxed{5}$	6	
4	4	5	6	7	8

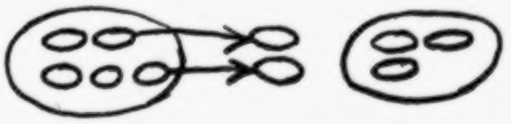
Under the operation of addition, 3 and 2 in either order yield the sum 5. The inverse operation indicates 5-3 = 2 and 5 - 2 = 3. We may assume that for every addition fact students will know one other addition fact (because addition is commutative), and they will also know two subtraction facts by performing the inverse operation.

3 + 2 = 5	5 - 2 = 3
2 + 3 = 5	5 - 3 = 2

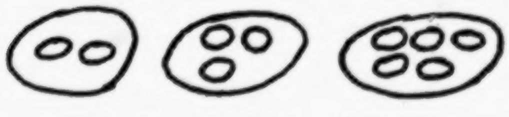
The addition and subtraction facts should be demonstrated by the use of objects so that children will see this concretely.



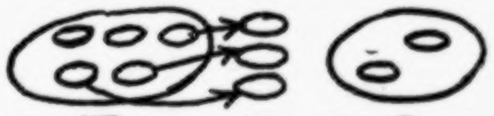
3 + 2 = 5



5 − 2 = 3



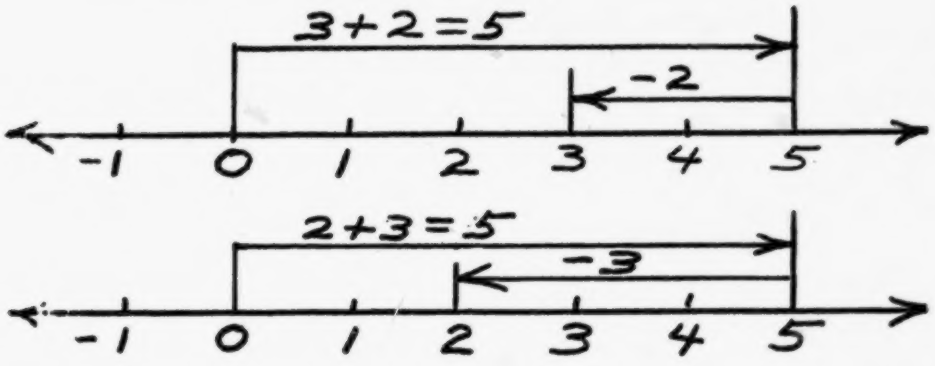
2 + 3 = 5



5 − 3 = 2

Demonstrations may be done with any suitable counters. Students themselves should manipulate the learning aid and be encouraged to discover the facts. The quantity subtracted should be physically removed, then the concrete manipulation represented with an algorism. As the facts are developed they should be written on the chalkboard and discussed until students do not need to depend on the use of aids. It should be emphasized that the operation is being performed on numbers and that these are cardinal numbers of each set (the "how many" concept of numbers), not the concrete objects. We operate on numbers, not on things.

The number line can also be used to demonstrate subtraction as the inverse operation of addition.



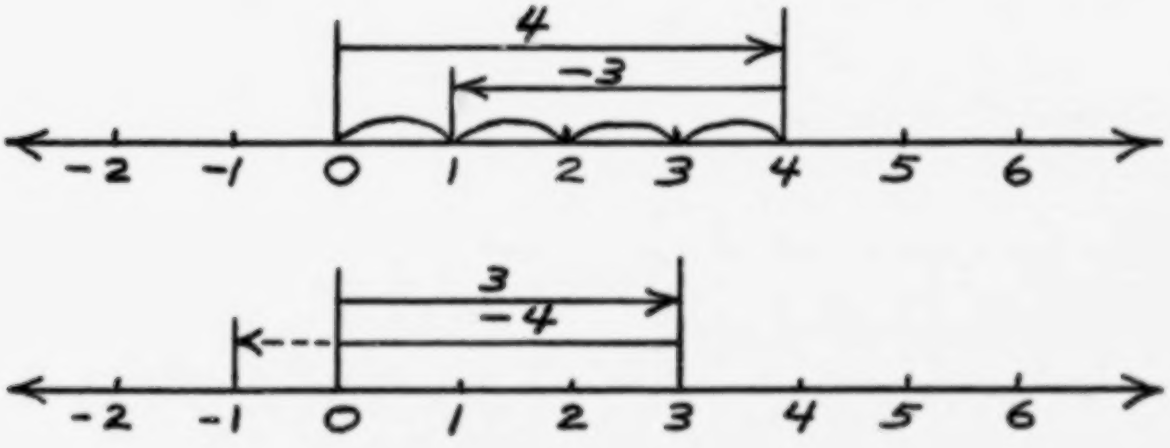
$$3+2=5$$
$$(3+2)-2=3$$
$$3+(2-2)=3$$

$$2+3=5$$
$$(2+3)-3=2$$
$$2+(3-3)=2$$

On the number line it can be seen that subtraction may also be thought of as identification of the missing addend. The expression 5 - 2 = ? may be written as 2 + ? = 5, indicating that one addend and the sum are known and it is necessary to find the missing addend. To compute 2 + ? = 5 on the number line, locate the point 2, then find the number of "steps" it requires to arrive at 5. Three steps are needed. It can be seen that it requires a distance of 3 units to arrive at the point 5. The missing addend is 3; the expression can now be written as 2 + 3 = 5. Other subtraction facts may be developed with the same procedure.

Subtraction is a binary operation (an operation performed on two numbers). It is not associative. The expression 5 - 3 - 2 has no meaning unless it is defined as to the manner of grouping. The expression 5 - (3 - 2) could be considered as 5 - 1 = 4. The expression (5 - 3) - 2 could be considered as 2 - 2 = 0. In other words, 3 - (3 - 2) ≠ (5 - 3) - 2 when the expressions in the parentheses are considered as representing one quantity.

In the expression 4 - 3 = 1, the sum is 4, the addends are 3 and 1. In the expression 3 - 4 = ? the sum is 3, the known addend is 4. The missing addend cannot be expressed with a natural number or zero. This can be shown on the number line as below.



Should children become familiar with the number line to the extent that they develop an intuitive understanding of the expression 3 - 4 = -1, they should not be discouraged.

The subtraction chart below will help to show that more numbers than the natural numbers and zero would be needed in order to make subtraction commutative. There is no result for the uncompleted portion of the chart unless negative numbers are employed. For example: The missing addend for the expression 3 - ? = 4 is found in the row opposite 3 (the sum) and the column beneath 4 (the known addend). It can be seen that there is no natural number or zero which will express the missing addend for the expression 4 ÷ ? = 3. This would be in the line opposite 3 (the sum) and beneath the column 4 (the known addend). The space is empty. In other words, 4 - 3 ≠ 3 - 4. When the operation is confined to the set of the natural numbers and zero, magnetic or flannelboard numerals can be used to build displays such as this:

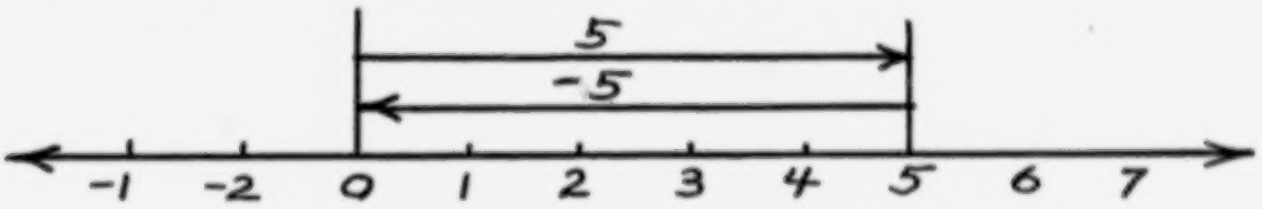
KNOWN ADDENDS

-	0	1	2	3	4	5	6	7	8	9
0	0									
1	1	0								
2	2	1	0							
3	3	2	1	0	?					
4	4	3	2	1	0					
5	5	4	3	2	1	0				
6	6	5	4	3	2	1	0			
7	7	6	5	4	3	2	1	0		
8	8	7	6	5	4	3	2	1	0	
9	9	8	7	6	5	4	3	2	1	0

MISSING ADDENDS

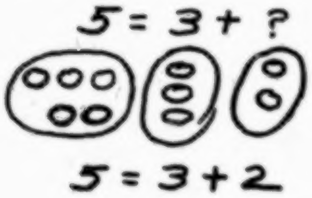
SUMS

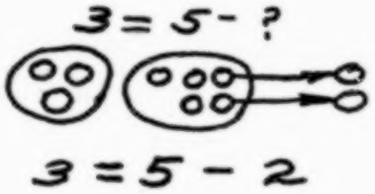
It can be shown on the number line that zero enables the performance of the operation of subtraction on "the same number" (zero enables us to subtract a number from itself: 5 - 5 = 0).



Learning aids furnish excellent opportunities for children to build good understandings of the order relationships of more than (>) and less than (<). Ask students to demonstrate by matching whether a set of 5 is more than or less than a set of 3, then to justify their conclusions and to verbalize their reasons. Use counters to demonstrate arithmetic expressions such as $5 > 3$. (Read "Five is greater than three.")

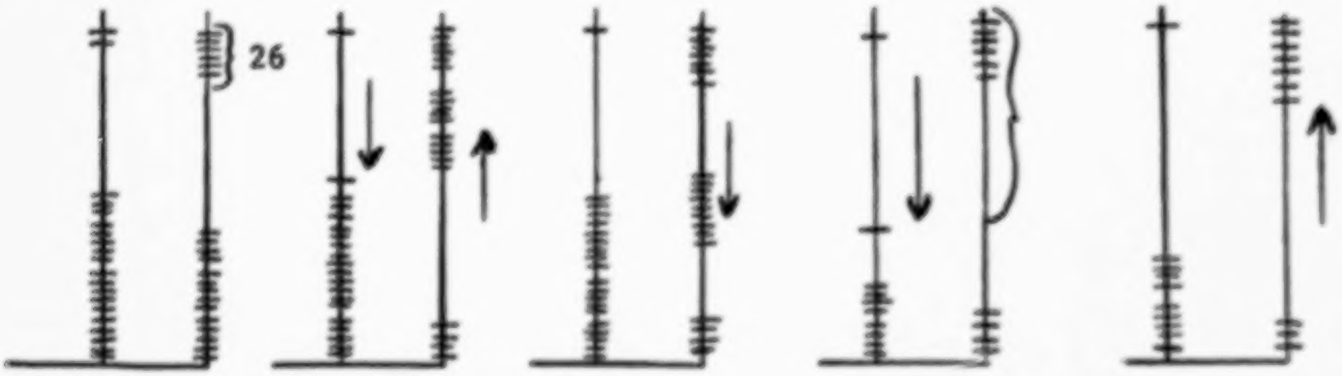
Students may show how many things make true sentences from such as the following:





Encourage students to describe the manipulation of learning aids, then to symbolize the operation. Reading many mathematical symbols can begin at a very early grade level and meaning be built as children learn to write the arithmetic expression which "tells" what is being done in the concrete manipulation. Use such expressions as: $5 + 3 \neq 8 + 1$, $5 + 3 < 8 + 1$, $n + 6 = 6 + n$. Help children to give reasons for their conclusions as to whether the expressions are true or false.

The abacus may be employed to demonstrate regrouping in subtraction examples. The example 26 - 9 may be computed in the two ways illustrated below. A twenty-bead abacus is the most convenient for such manipulation.



Exchange 1 ten for 10 ones. Subtract 9 ones. Result: 1(10) + 7(1) = 17

Subtract 1 ten. Replace 1 one. Result: 1(10) + 7(1) = 17

2 tens + 6 ones
- 9 ones

2 tens + 6 ones
-1 ten + 1 one (renames 9)
1 ten + 7 ones

The regrouping may be demonstrated as above. Children should experience many concrete representations before attempting the usual algorisms which do not present decomposition and regrouping in sufficient detail to develop a real understanding.

Flash cards which encourage students to employ the structural properties can be used. Cards could be developed like these below which emphasize understanding rather than the memorization of isolated facts.

15 SUM
7 KNOWN ADDEND
? MISSING ADDEND

$17 - 3 = ?$
$3 + ? = 17$

$14 - 3 = 11$
$11 + 3 = ?$
$3 + ? = 14$
$14 - 11 = ?$

$6 - 5 = 11$
$5 + ? = 11$
$11 - ? = 6$
$11 - ? = 5$

Children can and should be encouraged to develop many algorisms for computing. The operation of subtraction requires time to build basic understandings of decomposition and regrouping before the algorisms are attempted. This can be accomplished effectively with abaci, counting frames, magnetic counters for the chalkboard, flannelboard cutouts, and so forth. The learning aid should be chosen so that it is suited to the level of understanding of the children.

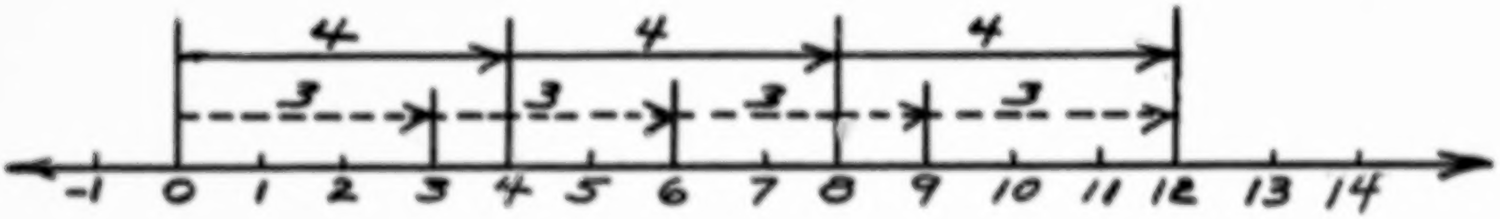
THE OPERATION OF MULTIPLICATION

In elementary mathematics multiplication is confined for the most part to the set of positive integers. Skill in computation must of necessity depend on a thorough knowledge of the multiplication facts. An understanding of the application of the structural properties in the development of the multiplication algorisms is also desirable. Concrete devices and learning aids can prove to be of much help in understanding the reasons for the various techniques employed in multiplication.

A thorough knowledge of positional notation is basic. Multiplication should be viewed as a distinct operation rather than as only an extension of the addition operation in which the addends are equal. Students need to develop sound understandings of the relationship between multiplication and division. Division then can be considered as the inverse operation of multiplication, in which the product and one factor are known and the missing factor is to be identified. The various division algorisms are then employed in order to find the missing factor. This may be computed as multiple subtraction, prior knowledge of the multiplication facts, and so on. In other words, if children known the multiplication facts it is not necessary to memorize a separate set of division facts.

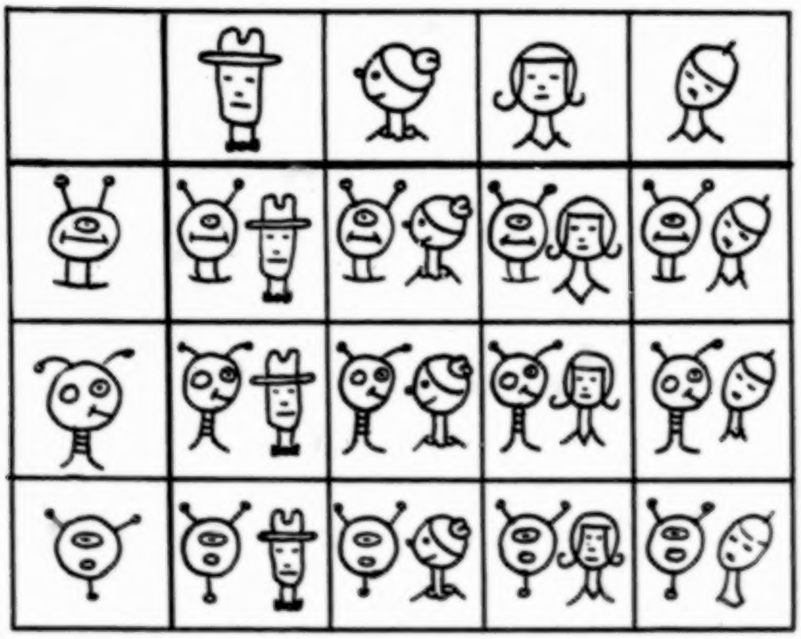
The various properties of multiplication should be applied to the operation (commutative, associative, distributive over addition, closure, and the properties of 0 and 1).

Learning aids can furnish children with many experiences to aid in understanding multiplication. One of the simplest is to consider multiplication as equal additions on the number line. The line below shows 3 x 4 = 12 and 4 x 3 = 12.



Representation of multiplication on the number line is limited to the size of the line and may become unsuitable for classroom presentations. When it is used, children should have their own number lines so that they may participate in the activity. This type of learning aid is perhaps best used with primary students.

We can think of multiplication as a pairing operation. The product of 3 x 4 may be considered as an array in which the members of a set of 3 are paired with a set of 4, resulting in a total of 12 pairings. Magnetic or flannelboard cutouts are excellent for this demonstration. Three things paired with 4 things yields 12 pairs.



Manipulative numerals can be used to construct an arrangement which shows the number pairs that yield the 100 multiplication facts. Table B shows the product under the operation for each number pair in Table A.

X	0	1	2	3	4	5	6	7	8	9
0	00	01	02	03	04	05	06	07	08	09
1	10	11	12	13	14	15	16	17	18	19
2	20	21	22	23	24	25	26	27	28	29
3	30	31	32	33	34	35	36	37	38	39
4	40	41	42	43	44	45	46	47	48	49
5	50	51	52	53	54	55	56	57	58	59
6	60	61	62	63	64	65	66	67	68	69
7	70	71	72	73	74	75	76	77	78	79
8	80	81	82	83	84	85	86	87	88	89
9	90	91	92	93	94	95	96	97	98	99

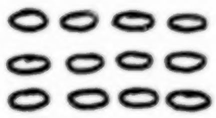
TABLE A

X	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

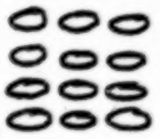
TABLE B

There are ten elements in the row 0 to 9 and ten elements in the column 0 to 9. There are 10 x 10 pairings. When the operation of multiplication is performed the 100 multiplication facts are yielded. It is evident that the crossed-off section of the chart is a mirror image of the other part. The order of the factors makes no difference in the product under the operation of multiplication. Therefore, with the exception of the "squared" numbers on the diagonal, the number of facts necessary to commit to memory is reduced by half.

Learning aids make suitable arrays to indicate the product when multiplication is performed on 2 numbers. A 3 by 6 array shows the product yielded by the operation of multiplication on 3 and 4. The two arrays also indicate that multiplication is commutative: 3 x 4 = 4 x 3.

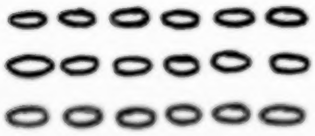


3 x 4 array yields
a product of 12



4 x 3 array yields
a product of 12

Magnetic counters or flannelboard devices are suitable for arrays. Let children demonstrate various "facts," then symbolize these. Ask that an array be made to show the product of 3 x 6 as in A.

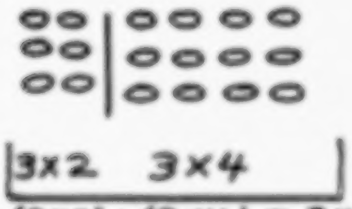


$3 \times 6 = 18$
A

$3 \times 3 \quad + \quad 3 \times 3 = 18$
B

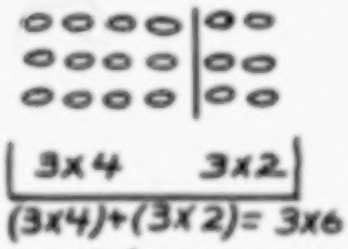
Ask students to arrange the array so that it represents (3 x 3) + (3 x 3) as in B. Then symbolize as 3 x 6 = 3(3 + 3) to show that multiplication is distributive over addition. Develop other combinations with student participation and experimentation until pupils are able to verbalize and to write the abstract idea without using aids.

Make a 3 x 6 array on the chalkboard and indicate the distributive principle with lines. Such arrays can be made with counters.



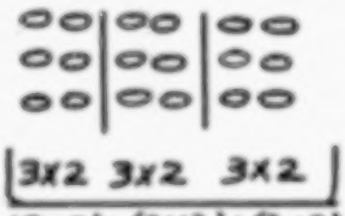
$(3\times2)+(3\times4) = 3\times6$

$3\ (2+4)=3\times6$



$(3\times4)+(3\times2)= 3\times6$

$3(4+2)=3\times6$



$(3\times2)+(3\times2)+(3\times2) = 3\times6$

$3[\boxed{2}+(2+2)] = 3\times6$

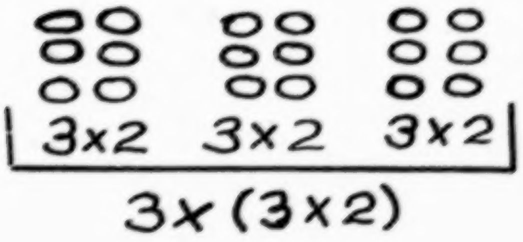
Let students experiment and develop arithmetic expressions to illustrate the concrete manipulations which they are performing.

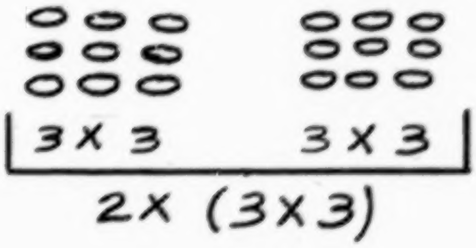
The associative property of multiplication is important in helping students to understand such expressions as 3 x 3 x 2 = ? which have little meaning to children unless the grouping is indicated:

$$3 \times 3 \times 2 = ? \quad (3 \times 3) \times 2 = 3 \times (3 \times 2)$$
$$9 \times 2 = 3 \times 6$$
$$18 = 18$$

Students should be encouraged to verbalize their thinking and to demonstrate with devices, then to participate in developing suitable algorisms for computation in multiplication.

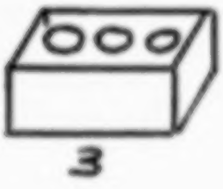
The grouping below can be shown with magnetic or flannelboard counters.

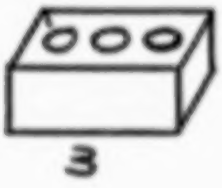


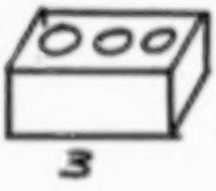


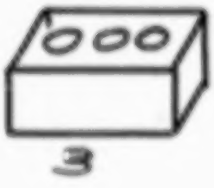
It should be shown that the expressions (3 x 3) x 2 may be thought of as 2(3 x 3), and (3 x 2) x 3 may be thought of as 3(3 x 2), because multiplication is commutative. The expressions (3 x 2) and (2 x 3) may be treated as representing one factor. When children understand the structural properties they are less likely to conclude that 3 x 3 x 2 is equal to 8 or 11, as is often the case.

It is often extremely difficult to convince youngsters that the product of any number (n) and zero (0) yields zero, and that the product of n and 1 is always n. A concrete representation of the product of any real number and zero and any real number and 1 may be demonstrated with learning aids. Arrange any number of empty containers. Place 3 counters in each.









The number 3 names the quantity in each container. There is a total of 4 x 3 (12) counters in all. Place in each container: first 3 counters, then 2, then 1, then none. Show that the total number of the counters is named by: 4 x 3 (12); 3 x 3 (9); 3 x 2 (6); 3 x 1 (3); and 3 x 0 (0). Help children to see that regardless of the number of containers, the measure of whose contents is zero (0), the total would still be 0.

Help students to abstractionize to the point that they can show 0 x n = n x 0 = 0, and 1 x n = n x 1 = n, by application of the commutative property of multiplication.

THE OPERATION OF DIVISION

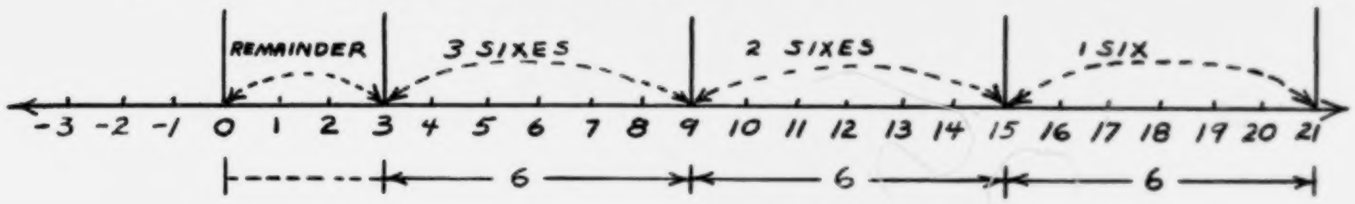
Activities which involve a comparison between groups or sets have been presented under the operation of subtraction. It is now necessary to employ the concepts of greater than (symbolized >) and less than (symbolized <) in order to develop an understanding of the algorism for long division. The symbol ≥ (greater than or equal to) and the symbol ≤ (less than or equal to) will also need to be employed.

Consider the division example 21 ÷ 6 = ? The quotient may be computed by finding the number of times 6 may be subtracted from 21 with or without a remainder. In other words, into how many 6's can 21 be regrouped?

The example can be demonstrated as in this algorism, which shows successive subtractions of 6's from 21.

6/21		
- 6	1	(Six subtracted 1 time)
15		
- 6	1	(Six subtracted a second time)
9		
- 6	1	(Six subtracted a third time)
3	3	(The sum of the 6's subtracted is 3. There is a remainder of 3 ones: 21 = (3 x 6) + 3.)

This can be shown on the number line as:



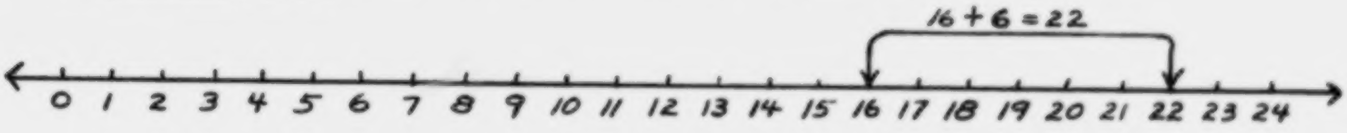
It is evident from the number line that the operation could be extended on to the left, passing zero, to include another "step" of 6. It should therefore be pointed out to students that this operation is performed on the set of positive integers.

The quotient may also be found by reference to the multiplication "facts." The student thinks: ? x 6 = 21. If the correct fact is known a solution can be completed. However, experimentation may produce any of the following misconceptions, all of which are perfectly true.

$$\begin{array}{r} 2 \text{ r. } 9 \\ 6\overline{)\,21} \\ \underline{12} \\ 9 \end{array} \qquad \begin{array}{r} 3 \text{ r. } 3 \\ 6\overline{)\,21} \\ \underline{18} \\ 3 \end{array} \qquad \begin{array}{r} 1 \text{ r. } 15 \\ 6\overline{)\,21} \\ \underline{6} \\ 15 \end{array}$$

It is evident that youngsters should realize that this algorism involves the concepts of >, <, =, ≥, and ≤.

We now restate the example of 21 ÷ 6 as a consideration of the greatest number of sixes contained in 21, or ? x 6 ≤ 21.



Reference to a number line shows that the number of 6's must total an amount less than or equal to 16, because the addition of 6 to 16 would yield an amount in excess of the dividend 21. We therefore state this as: 16 ≤ ? x 6 ≤ 21. (Read: "Sixteen is less than or equal to what (some number) times 6 which is less than or equal to 21.") Students should be encouraged to symbolize the thinking needed for the computation. However, complete mastery in writing an arithmetic sentence is not the goal here. The object is to help children express the problem and to recognize an acceptable quotient.

Consider the following expressions as ways of thinking about the division indicated:

Is 2 x 6 < 21? (Yes)
2 x 6 ÷ ? = 21 (The student finds that 2 x 6 + 9 = 21.)
Is 3 x 6 = 21? (The student finds that 3 x 6 + 3 = 21.)

An acceptable quotient is 3 with a remainder of 3.

If the student begins with a quotient > 3 he may think: Is 4 x 6 ≥ 21? Reference to the multiplication facts or to a number line shows that 4 x 6 > 21. Therefore, 4 may not be used in the quotient. The student tries: Is 3 x 6 ≤ 21? and proceeds.

The principle of considering ≥ and ≤ can now be utilized in a more complicated division example, 436 ÷ 12. Thinking of this as decomposition of the set of 436 into subsets of 12's, we may compute it as a successive subtraction process.

```
12/ 436
   -120      10 twelves (10 x 12 = 120)
    316
   -240      20 twelves (20 x 12 = 240)
     76
   - 72       6 twelves (6 x 12 = 72)
      4      36 twelves  (A total of 36 twelves is contained in
                          436.)
                         (There is a remainder of 4 ones.)
```

Checking: $436 = (36 \times 12) + 4$

With the familiar algorism 12/ 436 the student may now think: What is the greatest number of 12's contained in 436, or ? x 12 + ? = 436. The remainder may or may not be zero.

```
     36  r. 4
12/ 436
   -360      30 twelves  (The digit 3 is placed in tens' position.)
     76
   - 72       6 twelves  (The digit 6 is placed in ones' position.)
      4                  (There is a total of 36 twelves in 436.)
                         (There is a remainder of 4 ones.)
```

Checking: $436 = (36 \times 12) + 4$

This algorism may be shortened to the following:

```
     36  r. 4
12/ 436
   -36      3 x 12  (The zero is omitted.)
     76             ("Bring down" the 6.)
   - 72
      4
```

If students have been exposed to a thorough understanding of the concepts of positional notation, less than or equal to (≤) and greater than or equal to (≥), they are better prepared to cope with the operation of division as performed on the set of positive integers with various algorisms. The emphasis should be on understanding the reasons for the format of the algorism rather than on simply duplicating it. This understanding is developed by beginning, not with the memorization of "long division facts," but with the acquisition of a thorough knowledge of the structural properties and the system of numeration.

The distributive property of division over addition could be illustrated in the example 12/ 436 by renaming the dividend (product) 436 as 360 + 72 + 4 and indicating division by 12 in some manner as:

$$12\overline{)\,360 + 72 + 4}$$

By performing the division operation we find that 360 divided by 12 yields 30; 72 divided by 12 yields 6; and the remainder is 4. (Combining: 30 + 6 = 36 r. 4) The missing factor is 36 and there is a remainder of 4. We can see that 436 is not exactly divisible by 12.

A more suitable expression for the example might be:

$$\frac{360}{12} + \frac{72}{12} + \frac{4}{12} = 30 + 6 + \frac{4}{12} = 36\frac{4}{12}$$

It is necessary not that students engage in such intricacies of computation but that they understand the distributive property of division over addition. This can be more easily demonstrated with an example in which the dividend is a multiple of the divisor: $12\overline{)\,36} = 12\overline{)\,24 + 12} = 2 + 1 = 3$. In this example the dividend (product) is exactly divisible by the divisor (known factor) and the missing factor is easily identified.

Manipulative numerals are particularly well adapted to demonstrating the distributive property. In the algorism $12\overline{)\,436}$ it is possible to use manipulative numerals, remove the dividend physically, and substitute 360 + 72 + 4. Encourage youngsters to experiment with different expressions for the dividend.

"Dividing and sharing" things is a real-life situation for primary children. Their world is largely parts of things. The division operation should not be excluded from their school experiences. Many learning aids can be employed to develop beginning understandings of the operation of division. This can be a worthwhile and interesting discovery process. Learning aids such as counting frames cause less confusion than devices constructed with removeable beads or sections.

Choose a quantity within the experience of youngsters such as 6. Represent 6 on the counting frame.

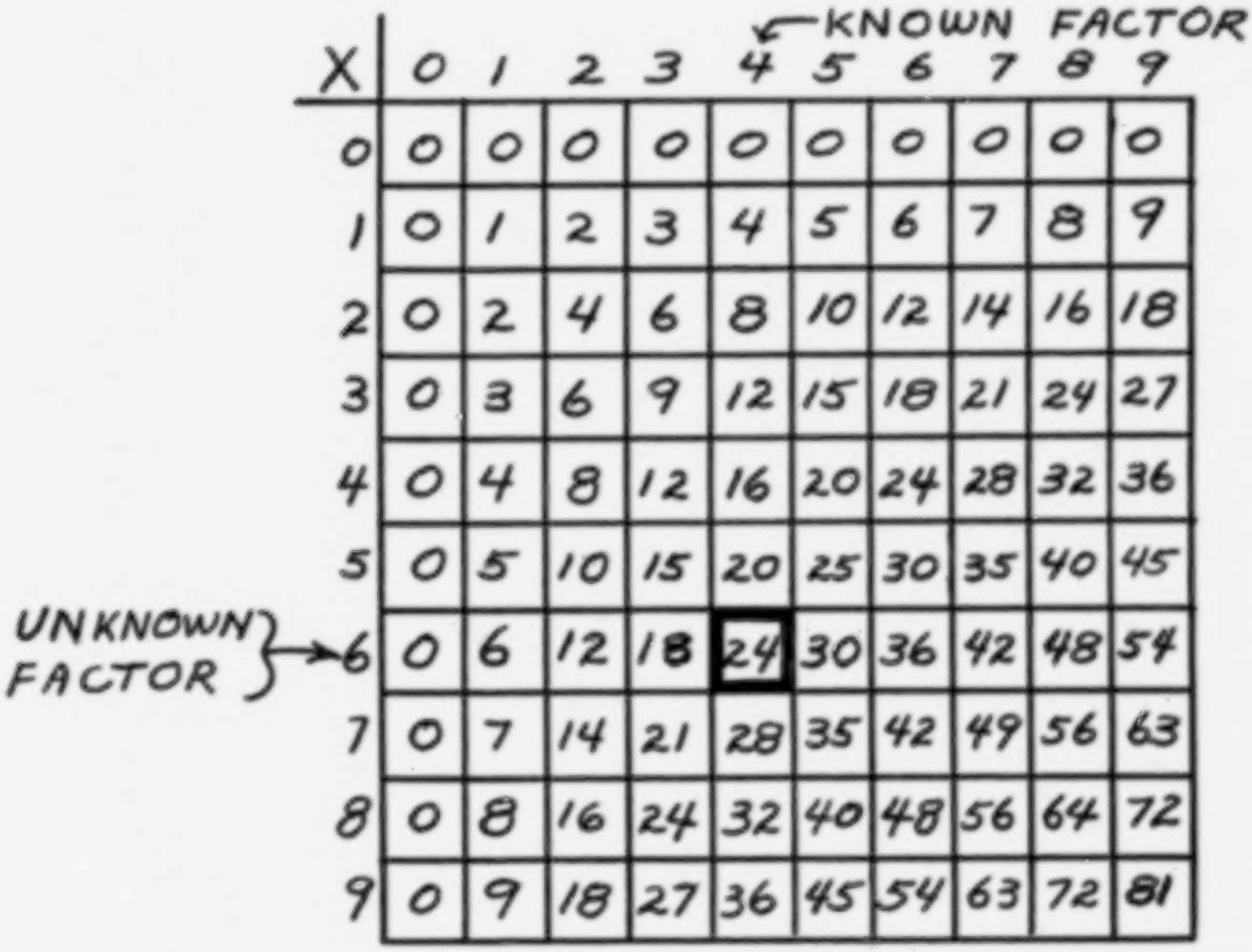
Ask such questions as:

Who can show 6 on the counting frame?
Who can divide 6 in half? (Some children will do this mentally; others will slide beads apart until they have two equal groups.)
How much is half of 6?
Does this mean 6 divided by 2? Why do you think so?
How can we be sure if 3 is really half of 6? (Let children use the counting frame to demonstrate their thinking.)

Change the number to 7. Ask the same questions. Help children to discover that "we have no number to show half of 7", or that "it would be 3 and a half". Encourage invention of a way to write three and one-half. Encourage students to use the device for experimenting at increasing levels of difficulty.

When these ideas have been developed, ask such questions as: If half a number is 3, what is the number? How could we move the beads to find out? (Let students experiment with the counting frame.)

The counting frame or counters can also be used to develop concepts of quantity. Make a class activity of discovering how we might show 2 x 3 on the frame or with counters. Discuss whether this means 2 threes. When the product is found and represented on the frame, help children to discover that 2 x 3 = 3 x 2, and that 6 ÷ 3 = 2, 6 ÷ 2 = 3.


← KNOWN FACTOR
UNKNOWN FACTOR →

X	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

Chalkboards which have been divided into grids suitable for making an arrangement of the multiplication facts can be used to demonstrate the relationship between multiplication and division. If the product and one factor are known, the operation of division will yield the other factor. On the chart above, 24 is identified as the product of 6 and 4.

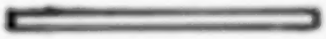
If the product 24 and the factor 4 are known, division will yield the unknown factor, 6. It is therefore evident that if a student knows that 4 x 6 = 24 he will know three other facts: 6 x 4 = 24 (multiplication is commutative), 24 ÷ 6 = 4, 24 ÷ 4 = 6 (identification of the unknown factor). Classroom activities involving the use of the structural properties can do much to reduce the task of memorizing the necessary facts and to increase the understanding basic to computation.

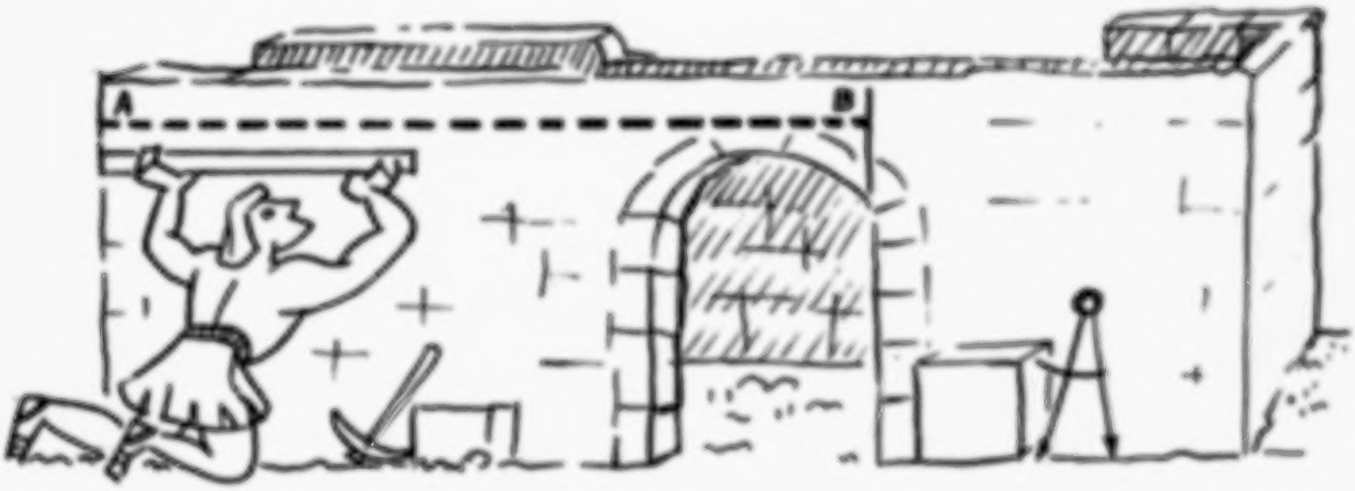
PART V: FRACTIONS

HISTORY AND DEVELOPMENT

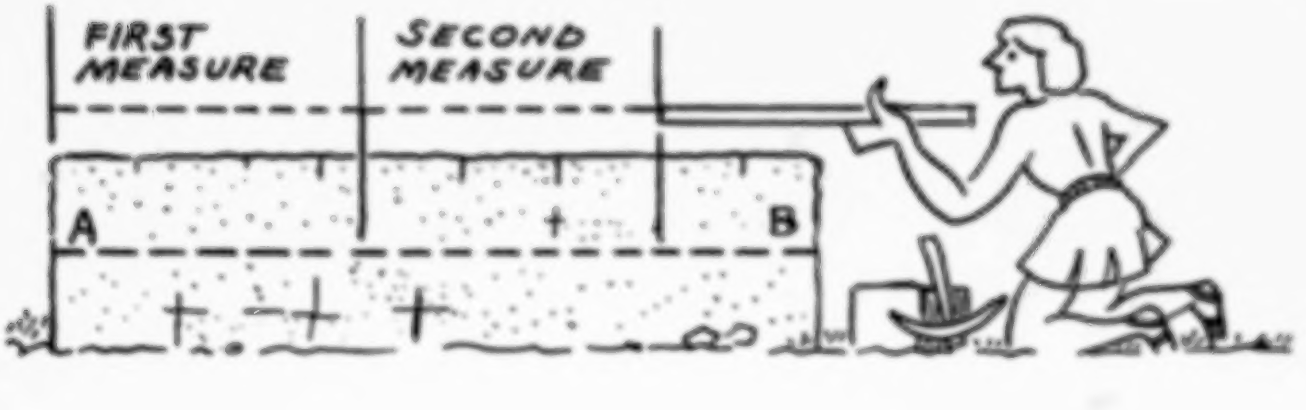
Students who have had the opportunity to understand the structure and logical development of the number system often find that fractions are not full of mystery and difficulty, but rather are useful and can be manipulated easily.

Like so many other innovations in mathematics, fractions probably were not created until man found a specific need for some method of expressing a part of a whole. Even though it is impossible to identify the exact beginning of fractions, the first use was doubtless in measurement. We might assume that ancient man may have been confronted by a situation in which he needed to measure the distance represented from A to B, and that the only thing he had to measure it with was a stick this long:

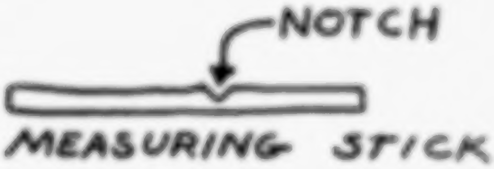




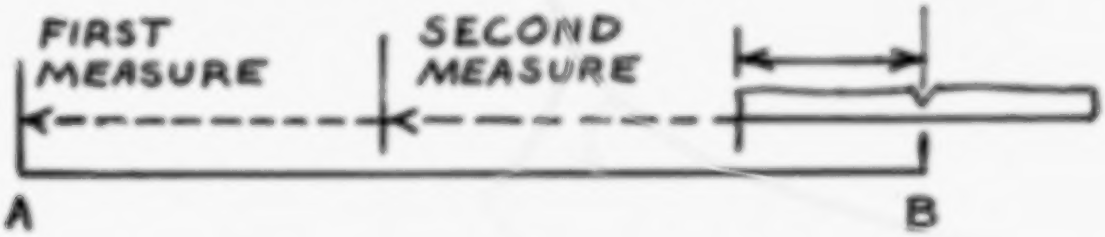
The distance could be measured in "sticks." When the man began to measure he found that the measure of the distance from A to B could not be described in whole "sticks". If he began at A and laid the stick down once, twice, and then tried again he discovered that it could be laid down twice with some distance left over that was not the same length as his measuring stick. The problem remained to name that distance which lay beyond the point where the second measure ended.



It then seems obvious that the measure of AB could be described in terms of "sticks" and "parts of a stick." The whole stick could be divided into sections or segments, and this would help to describe more accurately the distance from A to B. The important question was: How should the stick be divided so that it could be used to describe the measure of the line? It appeared that the part of stick left over on the third measure was about the same as the part of stick that had been used. Therefore, a notch cut in the stick as near the middle as possible could be used to describe this part-of-a-stick measure. Remember that the man had no name for the part.



The man now had a tool which he could carry around with him (move) and use to measure the distance AB. It could be reapplied to the distance, and used to describe its measure.

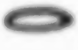
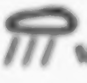
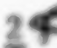
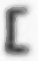


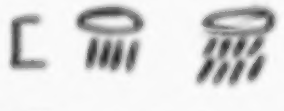
Now the problem remained to name that quantity which was represented by the part of a stick indicated by the arrow. If this could be done, then the measure of the distance AB could be stated in "sticks."

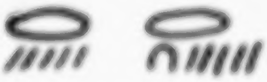
Many centuries passed before the man's action was described by mathematicians as a division process. The measuring stick would be considered as 1 whole length and the two equal parts symbolized by 1 divided by 2, later abbreviated to ½ and called "one-half." Then the distance from A to B could be accurately described as 2½ times the length of the measuring stick--in this case "2½ sticks".

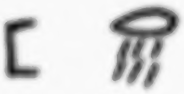
Students in the primary grades are entirely capable of going through this kind of discovery exercise and are able to find for themselves the significance of the fraction. The factual history of fractions can be briefly covered from investigations of early Egyptian and Roman mathematical writings. There are many reference materials which furnish rich and rewarding experiences that will help children to understand and appreciate man's struggle in designing a mathematics to describe the real world.

The English word "fraction" derives from the Latin **frangere**, meaning "to break." We have another word, "fracture," which has the same derivation. It seems that early cultures expressed parts of the whole by any method convenient to their symbolism. An example of the Egyptian method follows.

Since the Egyptians represented whole units by slashes, they invented a symbol that meant 1 part of the whole: . As /// represented 3, so  was used to represent one of three equal parts of the whole. We would write this as $\frac{1}{3}$. A special symbol was used for $\frac{2}{3}$  and for $\frac{1}{2}$  . Except for these two all fractions were expressed as unit fractions, fractions with numerators of 1. This symbolism resulted in some unusual ways of writing fraction numerals and of computing.

$\frac{7}{8} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} =$ 

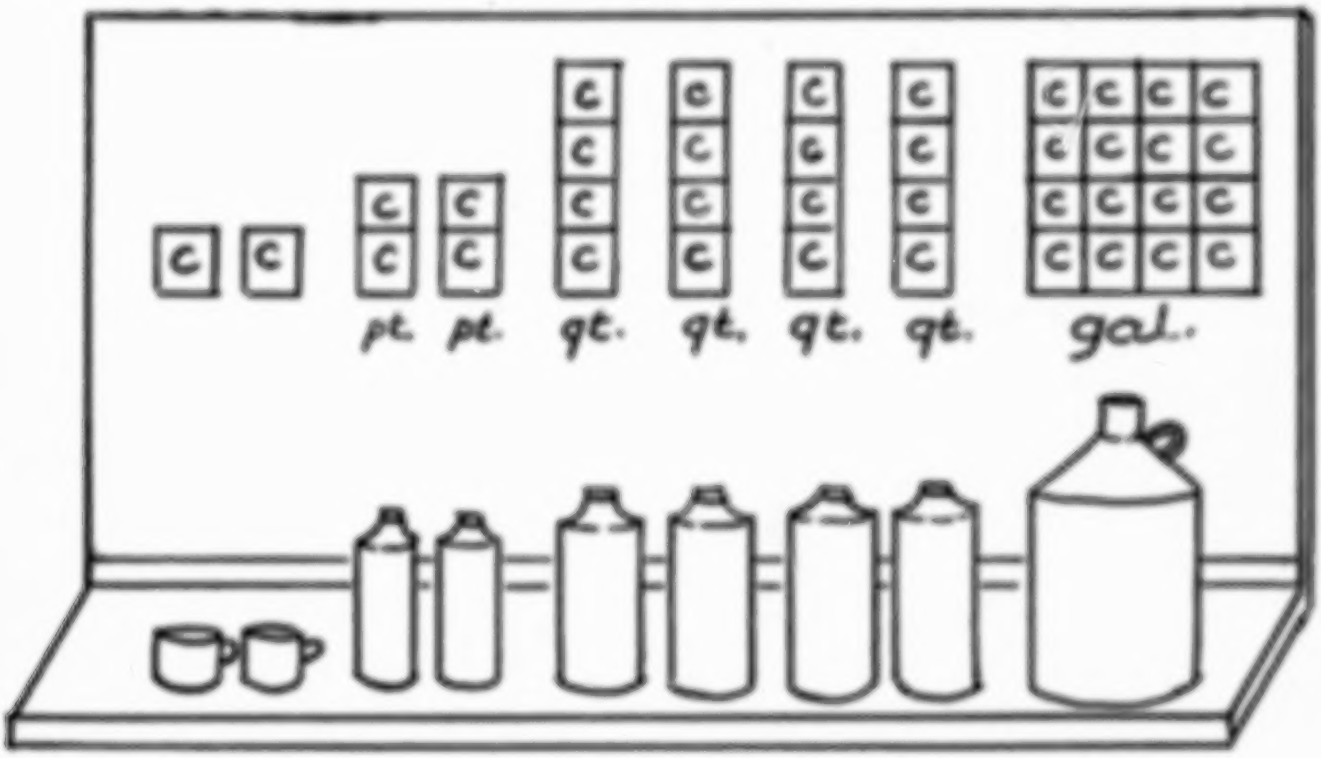
$\frac{4}{15} = \frac{1}{5} + \frac{1}{15} =$ 

$\frac{8}{12} = \frac{1}{2} + \frac{1}{6} =$ 

Activities using fraction numerals can be performed with a variety of measures. Students need to experience measuring rather than to observe demonstrations. Classroom activities should involve measures of length, area, volume, weight, time, temperature. After the concrete experience has initiated a good understanding, encourage students to develop their own algorisms for treating computation with whole and fractional measures. A supply should be available to students so that they may freely check their computation by using the actual device.

The representation of fractional concepts with concrete materials, such as culturally significant measures, fraction pies, fraction boards, and comparative charts, does not always assure that students will grasp the significance of the manipulation to the use of abstract symbols. It is not enough to memorize fractional equivalents or tables of equivalent measures. Students need to be involved in the activity of developing such relationships. Discovery and experimentation should be encouraged.

An arrangement like the one following can aid youngsters in using measures to develop fractional concepts by which they progress from the concrete representation to a graphic, then to the abstract symbol which can be employed in computation.



Students can manipulate these and similar devices in developing their own tables of equivalent measures, which will prove more meaningful than "ready-made" ones that are to be memorized. A suggested classroom procedure is given below.

1. Fill 2 cups, 1 pt., and 3 qt. with liquid. Empty 2 cups into 1 pt. Discuss the conclusions which can be drawn:

2 c. = 1 pt.
1 c. = ½ pt.

½ c. = ¼ pt.
Students will see this and go on.

Develop statements to be completed by students:

3 pt. = ? c.
6 c. = ? pt.

4 pt. = ? half cups
$\frac{12}{2}$ c. = ? pt.

Encourage students to develop others.

2. Empty 2 pints into the 1 empty quart. Discuss the conclusions which can be drawn:

2 pt. = 1 qt.
1 pt. = ½ qt.
½ pt. = ¼ qt.
½ pt. = 1 c.

1 c. = ¼ qt.

4 c. = 1 qt.
Students will go on.

Develop statements to be completed;

2 pt. = ? qt.

4 pt. = ? qt.

1 pt. $\frac{?}{?}$ = qt.

½ pt. = $\frac{?}{?}$ qt.

Encourage students to develop others.

Empty the 4 quarts into the gallon. Continue to draw conclusions and to develop comparative measures involving fractions. Continue similar activities with other units of measure. Investigate: (1)The suitability of the measuring unit for describing the measure of some object; (2) direct measures, such as the above; and (3) indirect measures in which some mechanical operation is used, as in clocks.

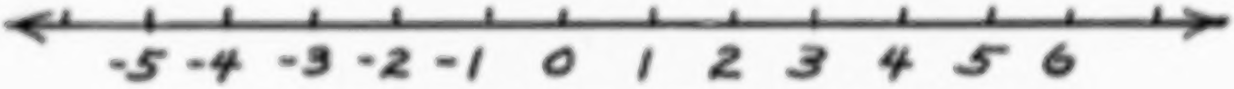
UNDERSTANDINGS NEEDED

One of the important uses of mathematics is to report characteristics of quantity in the real world. The most elementary question to be answered is "How many?" which requires a counting or mapping operation. Perhaps the next most common question is "How much?" which brings us to the very large area of measurement.

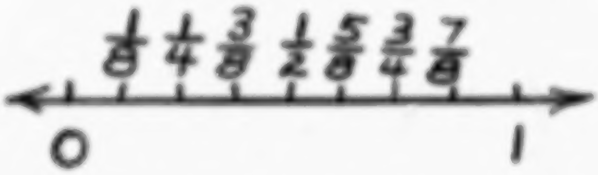
The historical development of our number system and its logical growth present an interesting parallel. First, man had a set of symbols (numerals) which were later given names and represent what we think of today as number. This process can be called the creation of the positive integers. Next, man realized that it would be necessary to divide quantities into smaller subdivisions than could be represented by only the integers; he therefore developed units signifying quantities of lesser magnitude than one whole integer. This process gave us the fractions, which should be referred to as the positive rationals. Following that came two more logical processes: 1) The extension of the positive rational system to incorporate the negative rationals; and 2) the extension of the positive rational system to include that conversion point where positives change into negatives and vice versa. We name that very important point zero.

As a result of this development, we are able to build a number line that has some very interesting characteristics. It contains the positive integers, zero, the negative integers, and many, many points which lie between the integers. All of these points go to make up the rational number system. Yet even though we are now able to identify them by their mathematical characteristics, we have not completed the system until we have included one more subset--the irrationals.

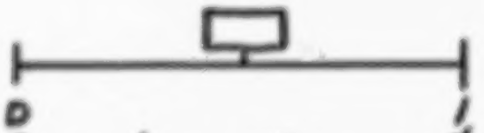
Consider the number line the way the child in the intermediate grades views it. He is familiar with a line with the integers and 0, such as this.



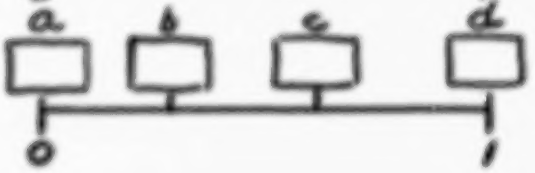
He also realizes that there are many points which lie between the integers, as depicted in the following diagram:



A good exercise at this point might be to provide the students with a great many number lines, each of which would be considered one unit long and have the fractional parts marked but not named with any fraction numeral. Examples would be:



(The student would write 1/2 in the ▭.)



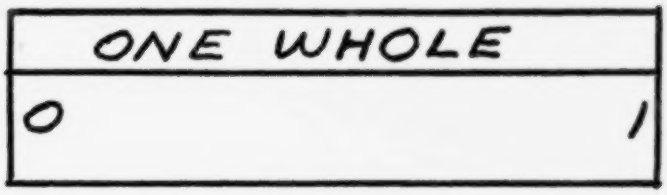
(The student would probably write 0 in box a, 1/3 in box b, 2/3 in box c, and 3/3 or 1 in box d.)

This type of exercise could be continued until fraction numerals including the twelfths were covered.

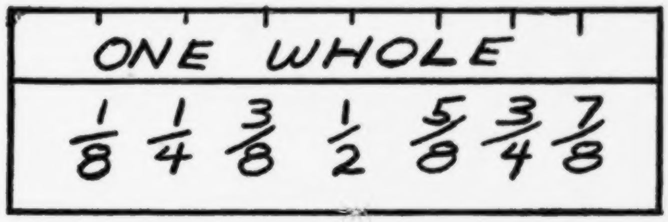
The next logical step would be to provide students with number lines one unit long which do not have the fractional values located by points. The student must approximate or devise other methods of locating the points and also naming them by writing the fraction numeral which they represent.

Students should be encouraged to use many methods to establish these understandings.

Examples (consider the halves, fourths, and eighths):



If the number line is made of flexible material the student can fold it in half a successive number of times. After being unfolded it will look like this.



Exercises of this type enable the student to compare values of various fractions and establish equivalence relationships between fractions. Repeated use of the unit number lines referred to above, with their fractional parts labeled, assists students in discovering facts like: $\frac{2}{8} = \frac{1}{4}$, $\frac{2}{4} = \frac{1}{2} = \frac{4}{8}$.

If the material from which the number line is made is not suitable for folding, the same effect may be achieved by having the student use a strip of paper equivalent in length to the unit line. Then the paper may be folded and used as a measuring instrument to locate the correct points. A piece of string, although more difficult to manage, may enable him to achieve the same results.

At this point the student should develop a clear understanding of the significance of the fraction numeral and be able to relate it to a physical quantity, such as a distance on a unit line. In addition, he should understand that the fraction numeral is composed of two separate numerals, one which is written above a line (which we shall call a division line) and one written below it. He also must understand that even though the fraction numeral is made up of two individual numerals, these must be thought of together as forming one symbol which names that part of an object or group of objects about which one wishes to talk or write.

The numeral written below the division line is called the denominator. This numeral really shows the number of equal parts, or subgroups, into which the object or group of objects has been divided. The numeral written above the line of division is called the numerator and shows the number of equal parts talked or written about.

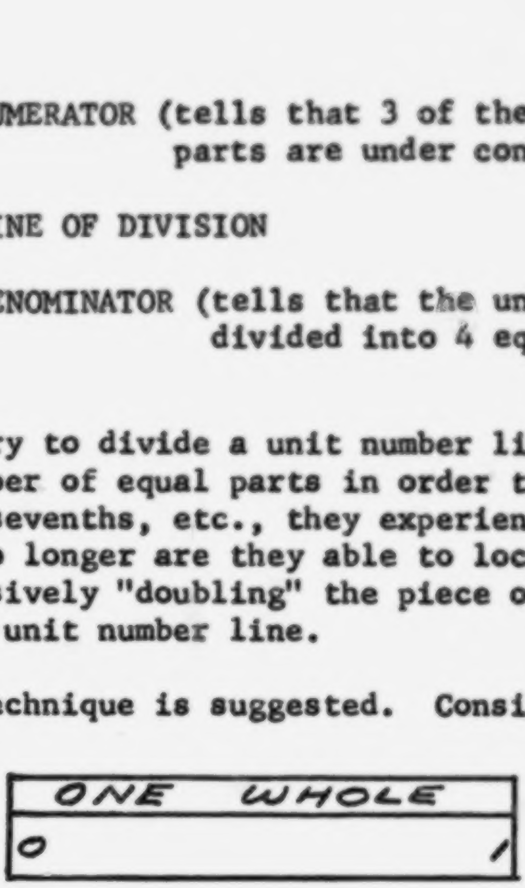
3 ←NUMERATOR (tells that 3 of the 4 equal parts are under consideration)

—— ←LINE OF DIVISION

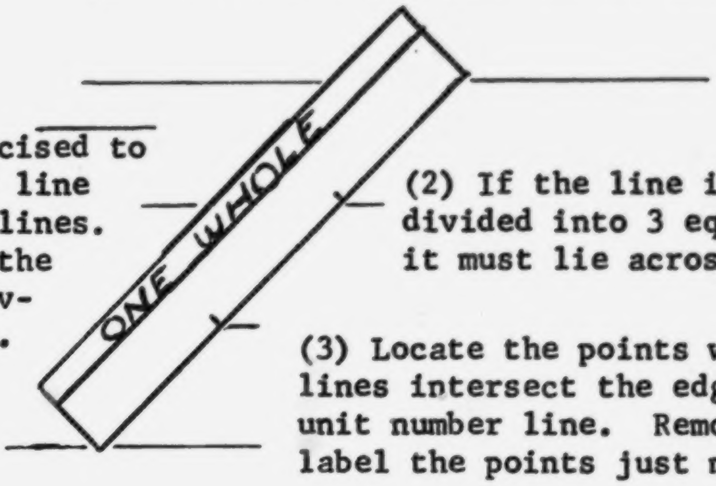
4 ←DENOMINATOR (tells that the unit has been divided into 4 equal parts)

When students try to divide a unit number line into 3, or 5, or 7, or any odd number of equal parts in order to find and label the thirds, fifths, sevenths, etc., they experience difficulty. They discover that no longer are they able to locate these points accurately by successively "doubling" the piece of paper equivalent to the length of the unit number line.

The following technique is suggested. Consider this unit number line:



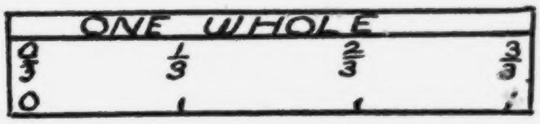
Assume that you wish to divide it into three equal parts. Select a ruled sheet of paper such as notebook or tablet paper and place the unit number line at an angle across the page, as in the illustration.



(1) Care must be exercised to place the ends of the line exactly on the ruled lines. Decide which edge of the unit line is to be divided into equal parts.

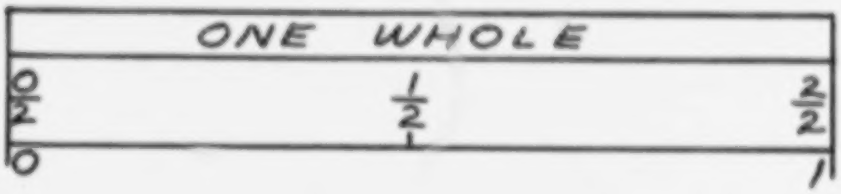
(2) If the line is to be divided into 3 equal parts, it must lie across 3 spaces.

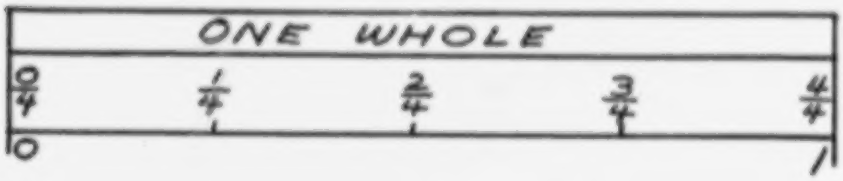
(3) Locate the points where the lines intersect the edge of the unit number line. Remove it and label the points just marked.

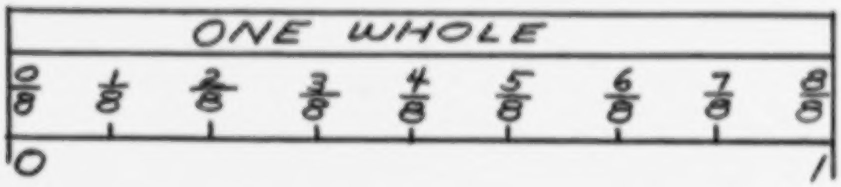


If the unit number line is to be divided into fifths, the number line has to lie across five ruled spaces on the paper; seven spaces for sevenths; and so on. The next step should be to use this knowledge to understand equivalent fractions and to develop procedures for finding common denominators.

The following set of unit number lines illustrates the notation necessary to compare equivalent fractions. Students need to understand that the unit line represents one whole. Beginning and ending points are indicated.







These unit number lines are to be compared superimposed one upon the other and manipulated, until the student is able to discover many facts such as:

(a) $\frac{2}{4} = \frac{1}{2} = \frac{4}{8}$

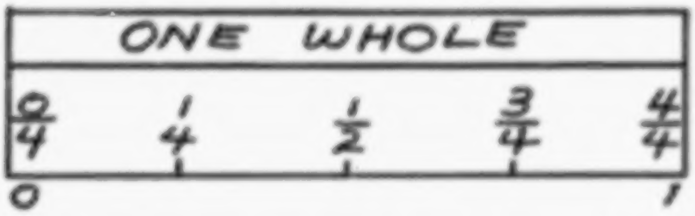
(b) $\frac{1}{4} = \frac{2}{8} = \frac{1}{2}$ of one half

(c) $\frac{3}{4} = \frac{6}{8} = 1\frac{1}{2}$ (halves)

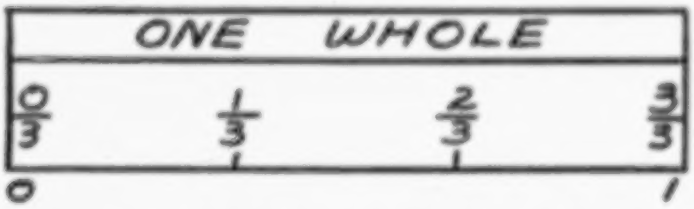
Procedures of this sort are to be followed until the student is able to generalize and hopefully do the exercise or provide answers to the questions without referring to the aids provided by the fraction number line. Exercises can be made more challenging by including unit number lines which have been divided into fractional parts whose denominators have no common multiples.

The next step is to undertake an illustration of the four fundamental processes with fractions. These can be very effectively illustrated with the use of unit number line as we consider addition--for instance, $\frac{1}{4} + \frac{1}{3} = ?$

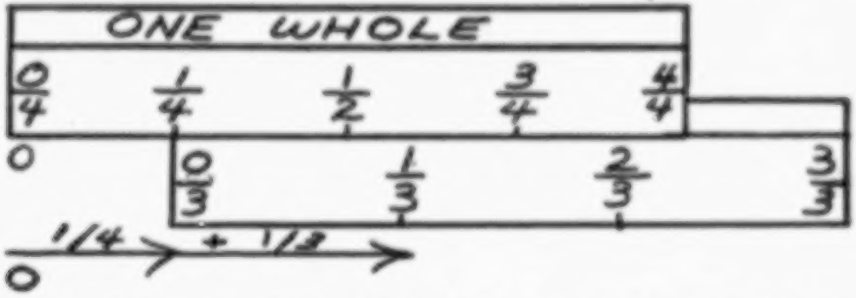
(1) Choose the unit number line divided into fourths and locate $\frac{1}{4}$.


ONE WHOLE
0/4
1/4
1/2
3/4
4/4
0
1

(2) Choose the unit number line divided into thirds and locate $\frac{1}{3}$.

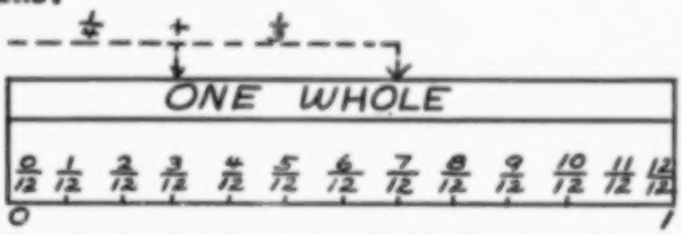

ONE WHOLE
0/3
1/3
2/3
3/3
0
1

(3) Place the number lines one beneath the other with the beginning of the one-third line opposite the point marked $\frac{1}{4}$ on the other.


ONE WHOLE
0/4
1/4
1/2
3/4
4/4
0
0/3
1/3
2/3
3/3
1/4
+ 1/3
0

(4) The problem now is to name the distance equivalent to the length of the two arrows $(\frac{1}{4} + \frac{1}{3} = ?)$.

(5) After experimentation the student will find that the unit number line divided into twelfths is the one which will name the combined length of the two arrows because both fourths and thirds can be expressed as twelfths.


1/4
+
1/3
ONE WHOLE
0/12
1/12
2/12
3/12
4/12
5/12
6/12
7/12
8/12
9/12
10/12
11/12
12/12
0
1

Therefore $\frac{1}{4} + \frac{1}{3} = \frac{7}{12}$, expressed as twelfths: $\frac{3}{12} + \frac{4}{12} = \frac{7}{12}$.

Many more exercises of this type should be experimented with by the student before he attempts to formalize and develop an algorithm for the addition of fractions. A variety of manipulative fractional learning aids lend themselves to exercises of this kind.

Extending this concept to include mixed numbers has the further advantage of being almost exactly like the procedure used on the integer number line for the discovery of addition with the integers.

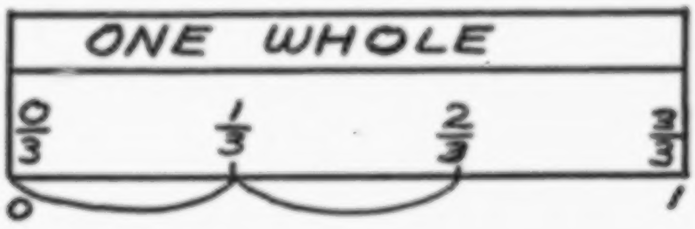
Since subtraction is the inverse of addition, it is experienced here in exactly the same manner as subtraction on the integer number line.

Multiplication, although similar to multiplication on the integer number line, is a little more difficult. Consider the following cases:

Case I. $2 \times \frac{1}{3} = ?$ or $\frac{1}{3} \times 2 = ?$

(Remember, multiplication is commutative; therefore, the order of the factors is unimportant.)

(a) Choose the unit number line divided into thirds.
(b) Locate $\frac{1}{3}$. The experimental exercise states that two steps of this length are to be taken. Begin at the end or the point $\frac{0}{3}$.

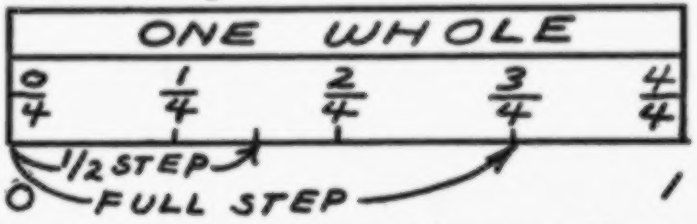


(c) The answer is, of course, $\frac{2}{3}$. $2 \times \frac{1}{3} = \frac{2}{3}$

Case II. $\frac{1}{2} \times \frac{3}{4} = ?$ or $\frac{3}{4} \times \frac{1}{2} = ?$

(a) In this case it would not make a great deal of difference whether one used the unit number line divided into fourths or into halves.

(b) Locate $\frac{3}{4}$. The problem indicates that we are to take a step which is only $\frac{1}{2}$ of this length.



(c) After trying several unit number lines we discover that half of $\frac{3}{4}$ can be named by using the unit number line divided into eighths and this unit is equal to $\frac{3}{8}$. We might also take a strip of paper equal in length to $\frac{3}{4}$, fold it in two, then name each half of its length by comparing with the eighths line.

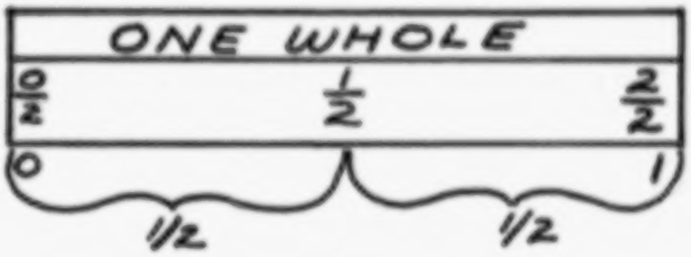
Division of fractions may be illustrated by the following cases.

Case I. $1 \div \frac{1}{2} = ?$

(a) Choose the line divided into halves.

(b) The division problem indicates that we begin with a distance of one unit and then determine the number of $\frac{1}{2}$ units contained.

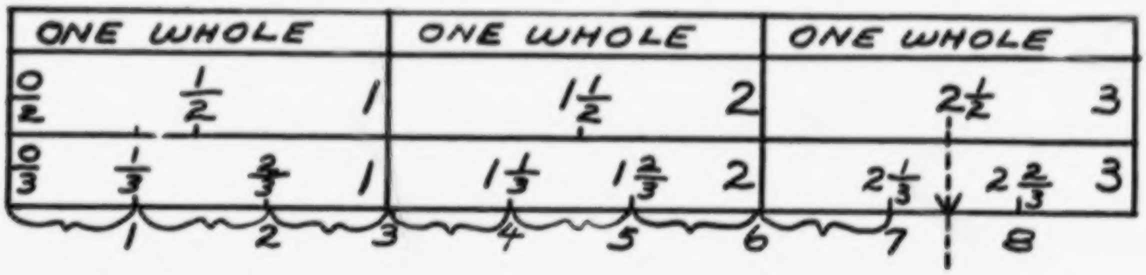
(c) Locate the point which is the same as one unit.



The number of one-halves contained is 2, or $1 \div \frac{1}{2} = 2$.

Case II. $2\frac{1}{2} \div \frac{1}{3} = ?$

It will be necessary to use 2 unit number lines plus part of another in order to express $2\frac{1}{2}$. These may be placed together as illustrated. From consideration of the diagram below we see that there are between 7 and 8 one-thirds contained in $2\frac{1}{2}$. The problem is, of course, to determine accurately the correct number. So we must find on what part of the distance between $2\frac{1}{3}$ and $2\frac{2}{3}$ the arrow falls:



Most students will estimate halfway and assume, therefore, that there are $7\frac{1}{2}$ one thirds in $2\frac{1}{2}$ ($2\frac{1}{2} \div \frac{1}{3} = 7\frac{1}{2}$).

Manipulation of learning aids does much to build the necessary understandings in developing the algorisms for computation. Other fractional representations such as fraction boards, fraction cubes, and fraction squares can be employed in helping students to acquire understandings.

PART VI. INFORMAL GEOMETRY AND MEASUREMENT

HISTORY

No one today would dare attempt to build a house, lay out a sidewalk, or build a road or bridge without using the basic ideas of geometry and measurement developed by the Egyptians many centuries ago.

Perhaps man's first use of the notion of quantity was to depict the size of something--most likely his possessions. As civilization progressed and man became more the master of his environment he began to build shelters and to take into cognizance more of the natural phenomena surrounding him. It required some notion of geometry and measurement to think about the apparent motions of the sun, moon, stars, clouds, and streams. As time passed, he began to identify himself with a certain area of land which he guarded carefully that he might ward off intruders. The art of building larger and larger structures grew as he extended his thinking to include more complicated notions of geometry and measurement. In the ruins of ancient cities in Egypt and other areas of the Middle East, we find the remains of many temples, pyramids, and impressive buildings which attest the ability of the early Egyptians, Babylonians, and later the Greeks.

After Alexander the Great's conquests, about 300 B.C., the famous Greek, Euclid, collected and organized old knowledge and created new knowledge of geometry. The axioms, postulates, and theorems systematized by Euclid stand today as one of the most coherent collections of carefully reasoned thought ever written.

A host of well-known early scholars, philosophers, and mathematicians delved into geometry with as much enthusiasm as did Euclid. Some of the more prominent were Appolonius, Archimedes, and Plato. Then came the Roman conquest, which ended the surge of creativity in mathematics fostered by the Greeks.

The next era of creative work in geometry occurred in the sixteenth and seventeenth centuries at the hands of famous scientists, philosophers, and mathematicians, such as Descartes and Newton. It was through their efforts that geometry grew into the important branch of mathematics that treats the shape and size of things.

THE LANGUAGE OF GEOMETRY

As we have discovered, the language of mathematics is very precise. This is particularly true in geometry. During the primary grades especially, great care should be exercised that children do not acquire incorrect notions of geometric terms. Too often a child thinks of a triangular area as a "triangle."

When fractional representations and geometric shapes are used on the chalkboard or flannelboard, attention should be given to correct terminology. In considering geometry and measurement for the elementary school it is difficult to separate the two concepts clearly. Therefore we shall consider them simultaneously.

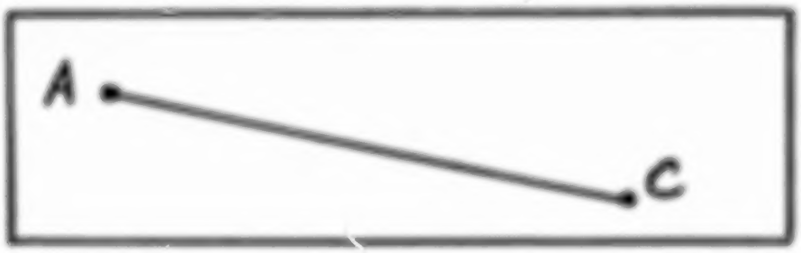
Probably one of the earliest notions a youngster has of geometry is the recognition of variety in the shapes of objects. He is also able to compare sizes. The "greater than" and "less than" ideas become a part of the real world of a very young child. Then he progresses to a stage in the kindergarten and primary grades when he recognizes the names of shapes, such as round, flat, square, triangular, circular, cylindrical,spherical, etc. It is well for any elementary teacher to have sets of geometric shapes for use in learning to identify objects by their area, volume, and contour. Sets of such materials are readily available from many sources. They are very good for making bulletin board displays, using as templates for student tracing, and as a source of material for recreational purposes like matching sets of triangles and sets of the same shapes.

Students do not go far with investigations of this nature before it becomes necessary to use more definite terms of comparison and a more complete descriptive language. Let us consider some of the essential terms.

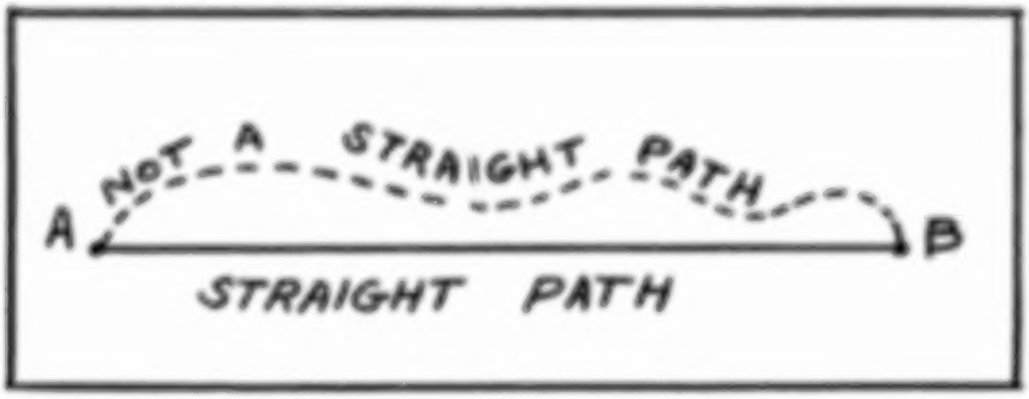
First we need a word to describe a particular position in space. Youngsters have this notion very early in their lives. They will need a little help in kindergarten and first grade in learning to identify the specific location ideas with the word "point". Points really have no size and cannot be measured. To say that they can even be seen is incorrect. However, in mathematical practice, it is true that a visible point is used to represent an invisible location. In geometry we have somewhat the same situation which we had with the number-numeral terms. Remember, we said that a number does not exist except as an idea. We create a symbol or numeral which represents a number. In geometry such things as points, lines, planes, and space are things or ideas we employ to aid in our thinking. On paper or in some other way we can create symbols or models which represent or stand for points, lines, planes, and space. Therefore, (.) is the symbol for a point. .A would be read as "point A".

Sets of points may be thought of as filling up a space. Think of the space shown below with a few of the points drawn in.

Remember that these are only a portion of the points which are in this space. For example, between A and C there are many, many points. We will represent the points by doing the following:

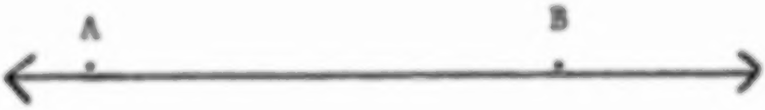


We have located a path between point A and point C. This is a straight path. It starts at point A and ends at point C. There would also be other paths one could follow in going from point A to point C such as the dotted path shown below:



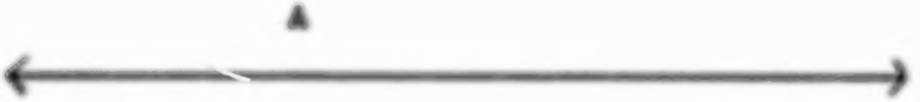
We had better name these paths so we can keep them separated in our thinking. The straight path is called a *line segment* and the path which is not straight, a *curve*. Line segments are symbolized by naming the beginning point and the end point, then drawing a bar over the capital letters which name these locations -- such as $\overline{AC}$. Remember, a line segment has a very definite beginning and ending. It is a "segment of a line"; therefore the word *line* itself takes on an important significance. Consider the following picture or model.

$\overline{AB}$ is a part of a longer line going across the page. The picture is an extension in both directions of "line segment AB" or $\overline{AB}$ and will be called a line. Another way to draw this line would be as follows:

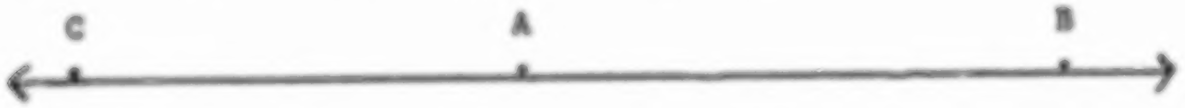


The arrow points at the end mean that the line goes on indefinitely in both directions. The symbol for the line would be $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$.

Any one point on a line will of course separate the line into two parts. Remember that the line goes on indefinitely in both directions and consider the following:



The correct way to think of this would be that point A has divided the set of points of which the line is composed into two subsets: (1) Those to the left of point A, and (2) those to the right of point A. A particular name for these two subsets of points would be helpful. The name which has been decided upon is a half line. The set of points which make up one of the half lines and the point A would be called a ray. We have created two rays. Rays are named by naming another point in each subset of points.



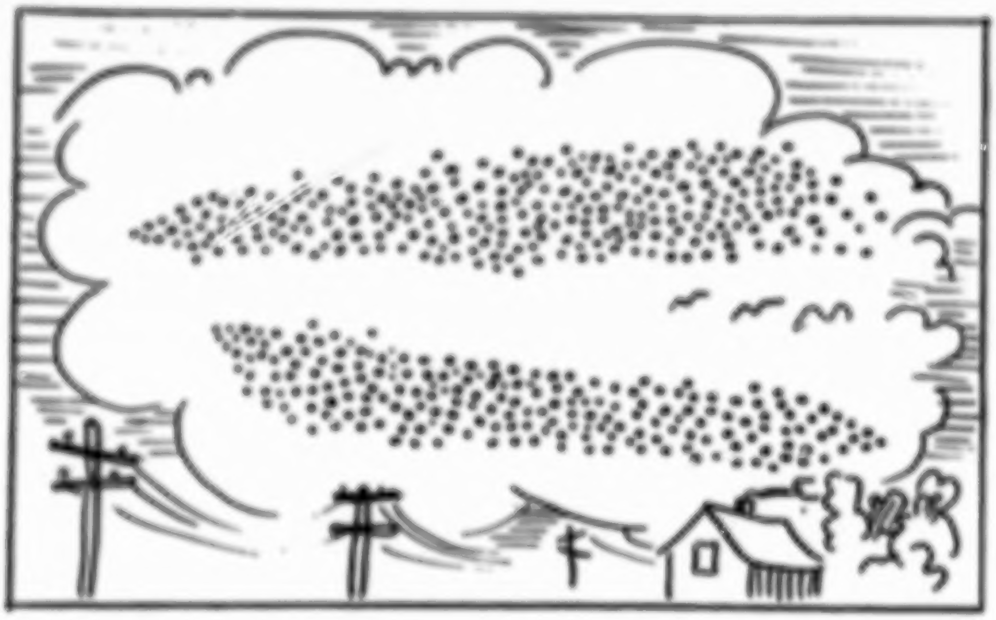
The points from which both rays extend is point A. We have one ray going from A to B and the other ray going from A to C. The symbols for the rays are written and read as follows: (1) The ray which goes from A to B will be labeled $\overrightarrow{AB}$ and read as "ray AB". (2) The ray which goes from A to C will be labeled $\overrightarrow{AC}$ and read as "ray AC."

Remember that a line extends in both directions without ending, a ray extends in only one direction without ending, and a line segment has two end points.

This characteristic of a line segment permits two different segments to be compared. Such a comparison of the lengths of two different line segments is called measurement.

Line segments which have the same length are said to be congruent. When two line segments are congruent this means that one of the segments, if it could be moved to cover the other, would match exactly. If two congruent segments represent the same set of points, then we say they are equal.

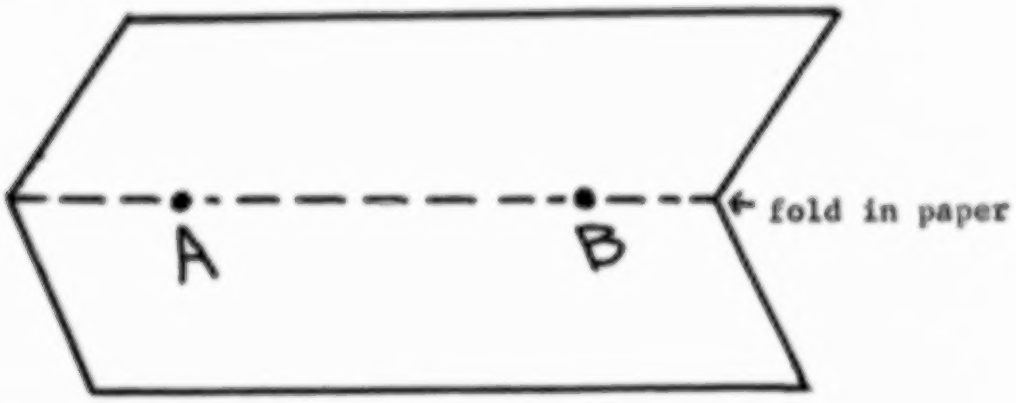
If we can continue our sets of points notion to include a set of points in space randomly arranged, but layered, we create another kind of geometric figure. For example, imagine that the following set of conditions could exist in which two sets of points make up two different planes.



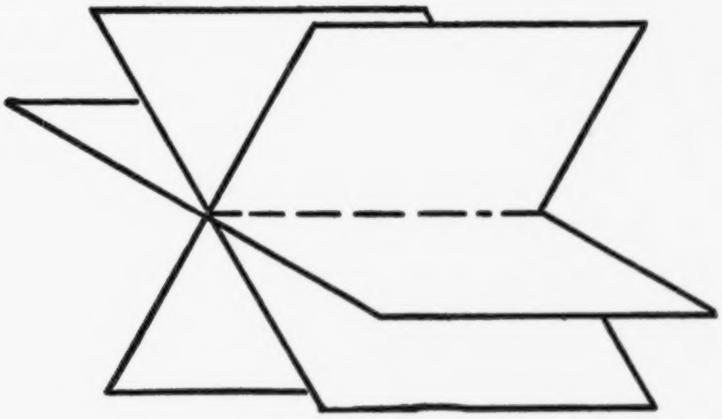
Youngsters in the primary grades can locate all sorts of models of planes in the classroom (i.e., tops of their desks, the chalkboard, the floor, etc.). Three concepts which must be clarified with the students regarding a plane are:

(1) A plane has a "flatness" characteristic.
(2) A plane is made up of a set of points and can be extended in all directions so long as it retains the flatness characteristic.
(3) A plane really cannot be drawn or constructed; we only construct, draw, or give examples of a model of a plane.

Consider a sheet of paper as a model of a plane. Fold the sheet as is shown in the following diagram:

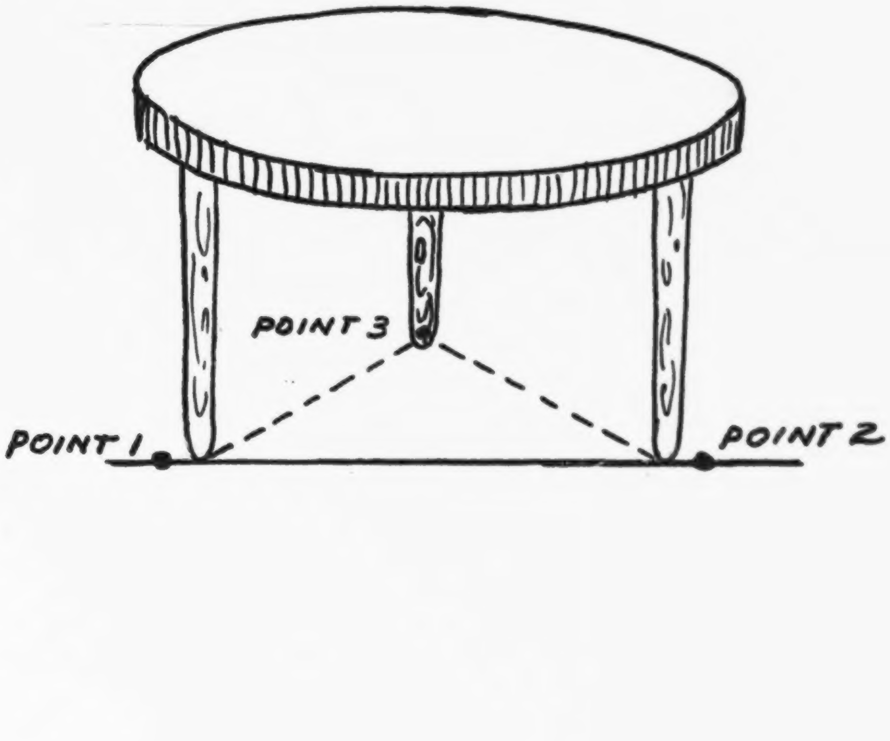


The two points A and B are on the crease. We have now formed a model for two planes. Line AB ($\overleftrightarrow{AB}$) is in both planes. Line segment $\overline{AB}$ is also in both planes. We can now formulate two important ideas about planes and a line. They are: (1) If a line contains two different points in a plane, it lies in the plane; and (2) two planes intersect or come together in a line. An extension of this idea demonstrates that many different planes could intersect in a line. A model of this situation would look like this:



Models of planes illustrating this concept are useful in the intermediate grades of the elementary school. They may be made from material such as chalkboard or plastic upon which points may be located, lines may be drawn, and so forth.

Students will have realized by this time that any two points in space will determine a line or line segment. If a third point in space is added (not on the line determined by the first two), then the three points are in a plane. Another way of saying this is: Three points not on the same line determine a plane. This condition suggests that a three-legged stool or table always rests solidly against a floor regardless of the fact that the floor may not be level or all the legs exactly the same length.

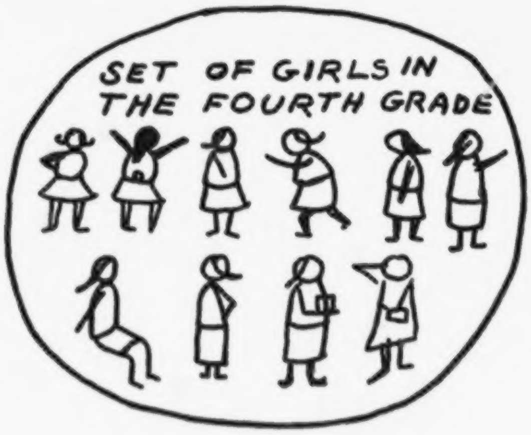


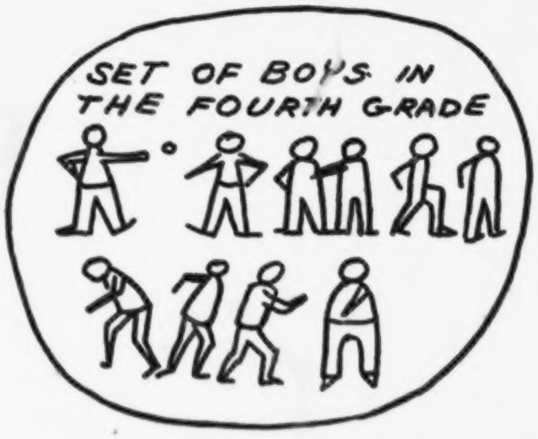
SETS AND THEIR APPLICATION TO GEOMETRY

In order to explain precisely some of the mathematical ideas of elementary geometry it is necessary to be familiar with the notion of sets and the symbolism used in dealing with sets.

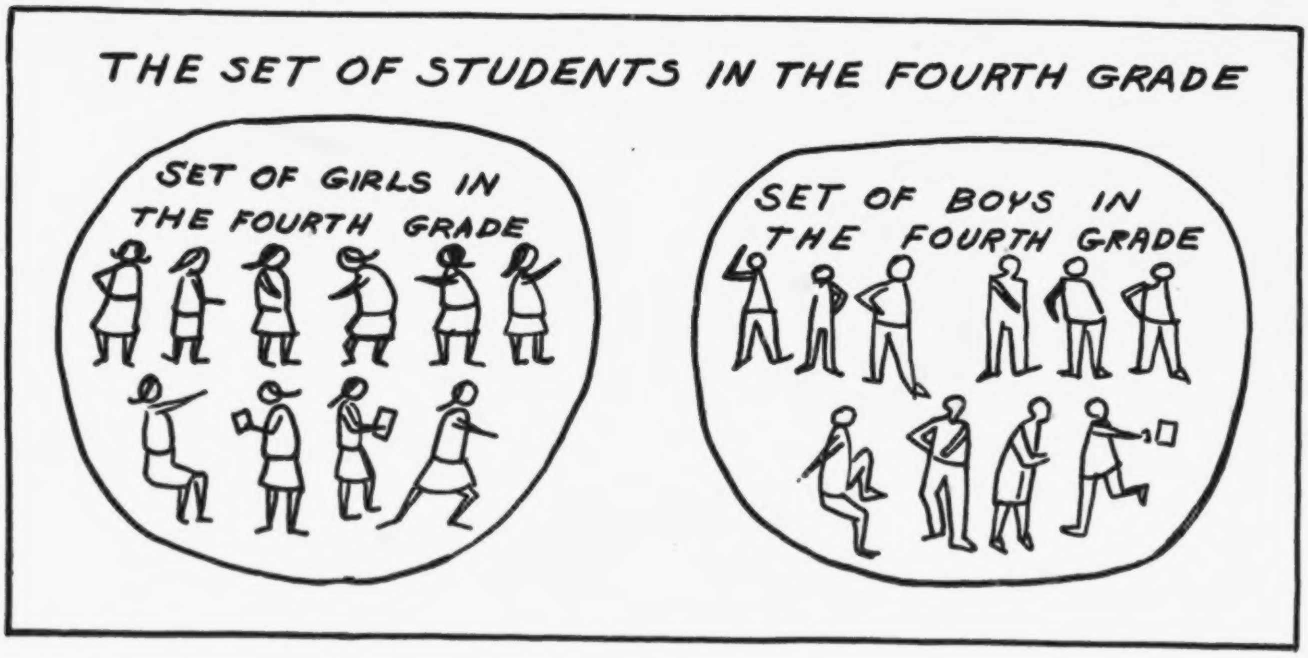
For our purposes, we will simply say that a set is a collection of well-defined objects. The objects which make up a set are called elements or members of the set. Examples would be a set of books, a set of dishes, the set of points on a line, the set of boys in the fourth grade, the set of girls in the fourth grade, and the set of students in the fourth grade.

Consider the last three examples for a minute and illustrate these with diagrams:


SET OF GIRLS IN
THE FOURTH GRADE


SET OF BOYS IN
THE FOURTH GRADE

We can now define another set such as the set of students in the fourth grade as follows:

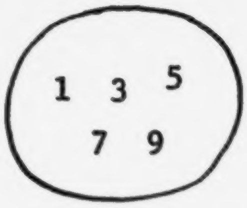

THE SET OF STUDENTS IN THE FOURTH GRADE
SET OF GIRLS IN
THE FOURTH GRADE
SET OF BOYS IN
THE FOURTH GRADE

The set of girls in the fourth grade and the set of boys in the fourth grade combined gives the set of students in the fourth grade. In this case the set of girls in the fourth grade would be a subset of students in the fourth grade. The set of boys would also be a subset of the students in the fourth grade. Capital letters are used to represent sets. The set of girls will be called set A, the set of boys set B, and the set of students set C. Then the set of students in the fourth grade would be called the union of set A and set B. There is a symbol for this notion of union. The symbol is $\cup$. We can now write our statement as follows:

$$A \cup B = C$$

The statement would be read: "The union of set A and set B is equal to set C." The union operation then directs us to find the set that includes all the members of the sets about which we are talking.

To give another example, we could define set A as the odd numbers from 1 to 10 inclusive. Then set A could be written in the form that was used with the boys and girls in the previous example, or it could also be written in another way.



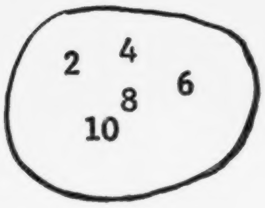
A

or

$\{1,3,5,7,9\}$

Set A

Define set B as the even numbers from 1 to 10 inclusive:



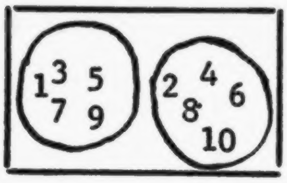
B

or

$\{2,4,6,8,10\}$

Set B

Now $A \cup B$ =



or $A \cup B$ = 1,2,3,4,5,6,7,8,9,10

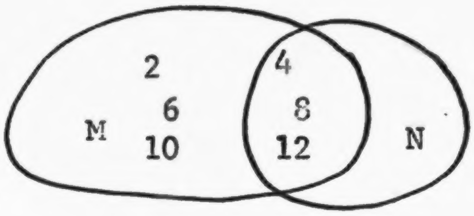
The union of two sets is known as an operation on sets.

There is one other operation on sets with which we need to be concerned in the elementary school.

Let us define set M to be the set of numbers from 1 to 12 inclusive that are evenly divisible by two. Then set $M = \{2,4,6,8,10,12\}$. Define set N to be the set of numers from 1 to 12 inclusive evenly divisible by 4; then set $N = \{4,8,12\}$.

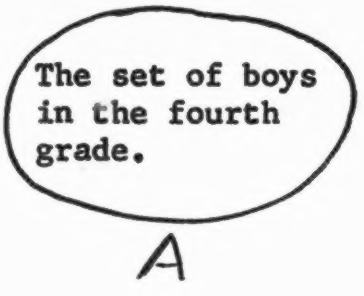
$$M \cup N = \{2,4,6,8,10,12\}$$

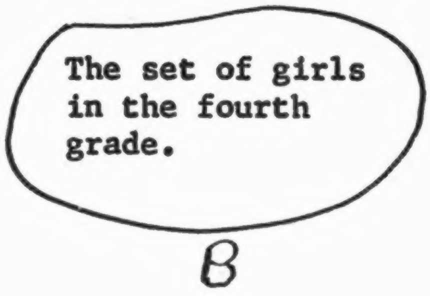
Sets M and N have some elements in common--namely, 4, 8, and 12. Under this condition, the name for this operation is intersection. The symbol for intersection is $\cap$. This symbol directs us to find the common members or elements of the two sets M and N. Another method of depicting this operation would be as follows:



$M \cap N$

The intersection of the two sets from the first example would be empty. That is, the intersection of the set of boys in the fourth grade and the set of girls in the fourth grade would not have any members because a student is always a boy or a girl. The situation would be depicted pictorially as follows:

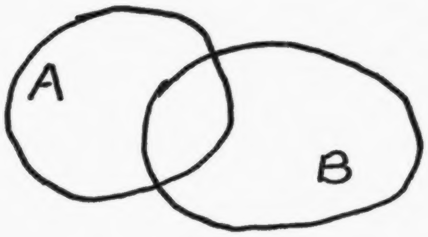




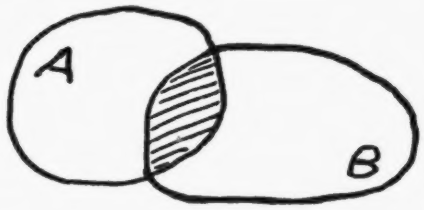
An empty set (a set that has no elements) is called a null set and is symbolized as ∅ (zero with a line through it). New we can write the intersection of set A and B as:

$$A \cap B = \emptyset$$

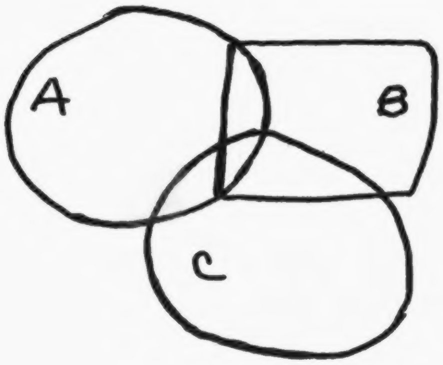
Students in the intermediate grades can understand these set concepts with no undue difficulty. They should have an opportunity to do many problems using set language and ideas. Overhead projections with overlays are an effective method of presenting some of the concepts of sets. Also, the bulletin board offers another possibility of adding interest in the classroom study by displaying set operations diagrammatically illustrated. Types of exercises which most students find appealing follow:



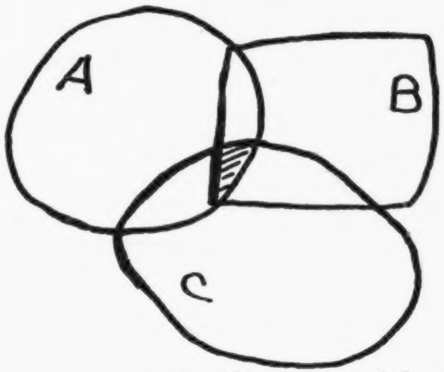
Shade $A \cap B$



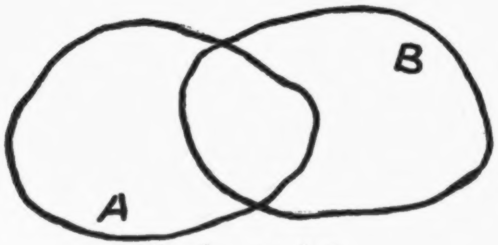
Student would shade



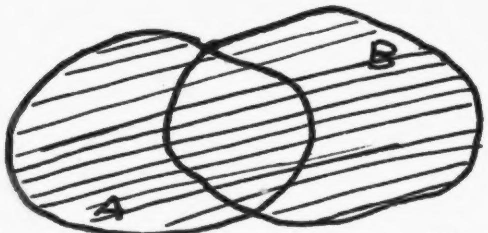
Shade $(A \cap B) \cap C$



Student would shade

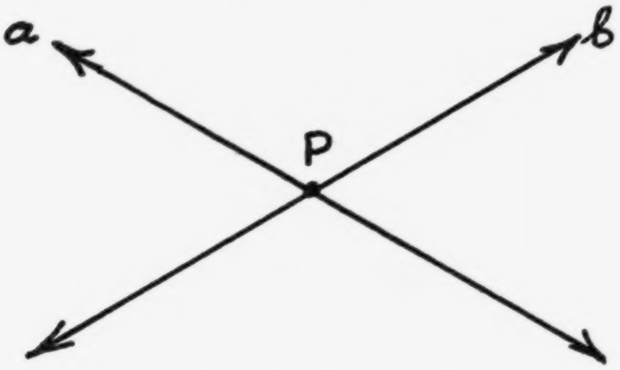


Shade $A \cup B$

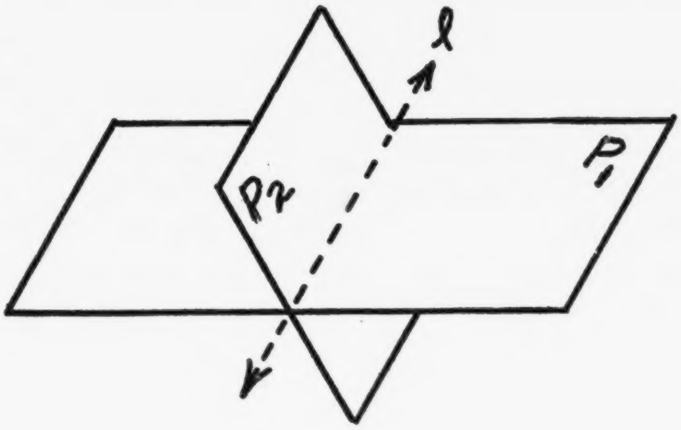


Student would shade

We find many applications for set ideas in geometry. For example, we have already talked about sets of points on a line, half—line, ray, line segment, etc. Notice the following examples:



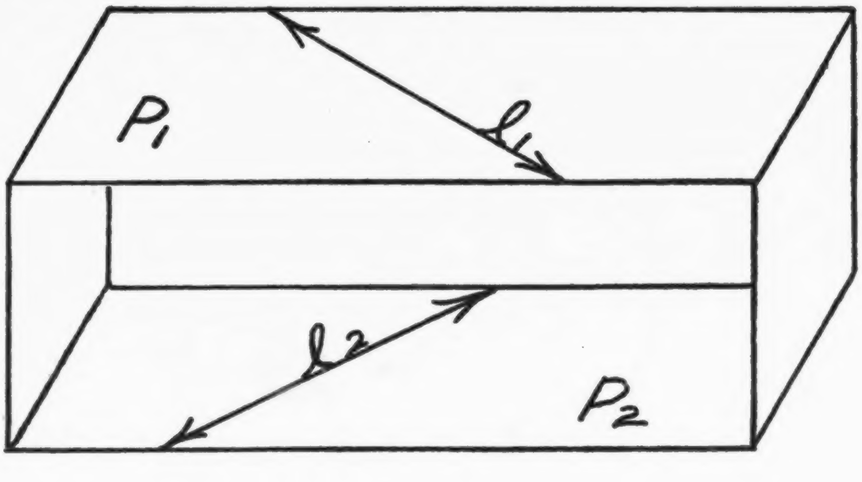
1. a and b are given intersecting lines. They intersect at point P. Let A be the set of points composing line a and B be the set of points making up line b. Then the following condition would be true: $A \cap B = P$



2. In the above diagram observe the two intersecting planes P_1 and P_2. P_1 and P_2 would intersect in line l. The following would be true:

$$P_1 \cap P_2 = l$$

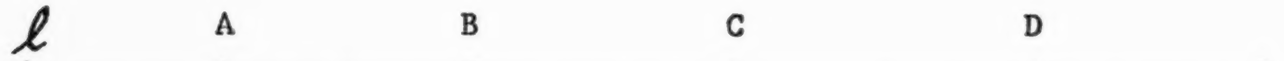
Read: "P one intersect P two equals line l."



In the diagram above, P_1 and P_2 are models of parallel planes, such as the floor and ceiling of the classroom. P_1 contains line ℓ_1 and P_2 contains line ℓ_2. Lines ℓ_1 and ℓ_2 will never intersect--Why? The set notation of this condition is:

$$\ell_1 \cap \ell_2 = \emptyset$$

3. Consider the following model of a line with points A,B,C, and D identified.



The points A,B,C, and D create the possibility of naming segments in line ℓ. The following are true:

$$\overline{AB} \cup \overline{BC} = \overline{AC}$$

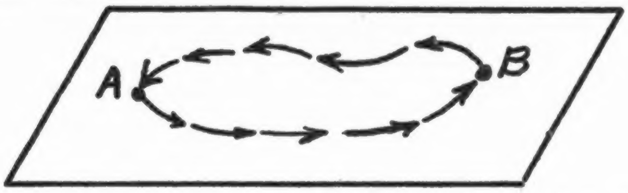
$$\overline{AB} \cap \overline{BC} = B$$

By this time it should be evident that set notions are very valuable tools to use in explaining geometric situations. Teachers will be able to employ many devices and models in the classroom to teach youngsters those ideas. For example, a model of a rectangular solid presents many opportunities to discover geometric concepts by naming lines, points, and planes and talking about the unions and intersections of the sets of points contained in each.

OTHER GEOMETRIC CONDITIONS

A few more concepts of elementary geometry are necessary before we can accurately describe the conditions which one encounters in measurement.

Examine the following model of a plane:



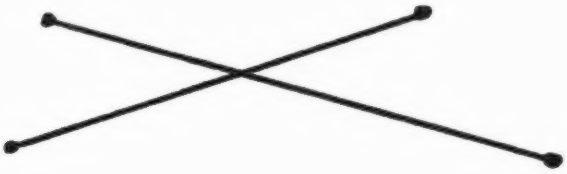
Points A and B are two specified points in the plane.

Choose any path from A to B, then choose a different path back to A. We have generated what is called a closed curve.

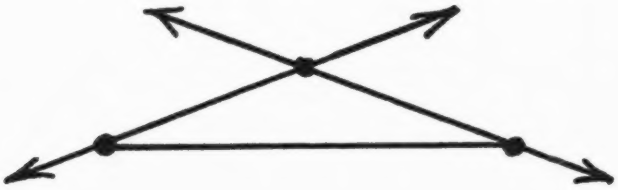
If a student walks to school by one route and returns to his home by another he has walked a path which could be described as a closed curve.

Remember, when we use the most direct or "straight" path between two points this is a line segment, but when any other path is used it is called a curve.

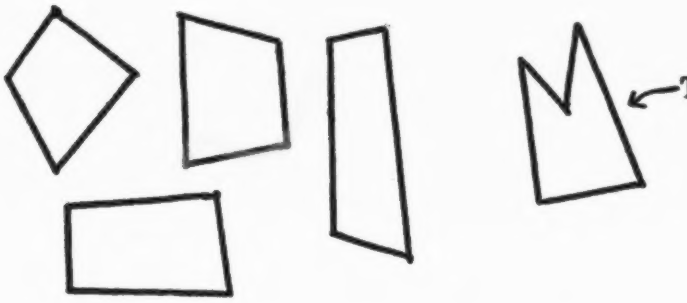
We cannot obtain a closed curve by using two line segments.



With three line segments it is possible to obtain a closed curve.

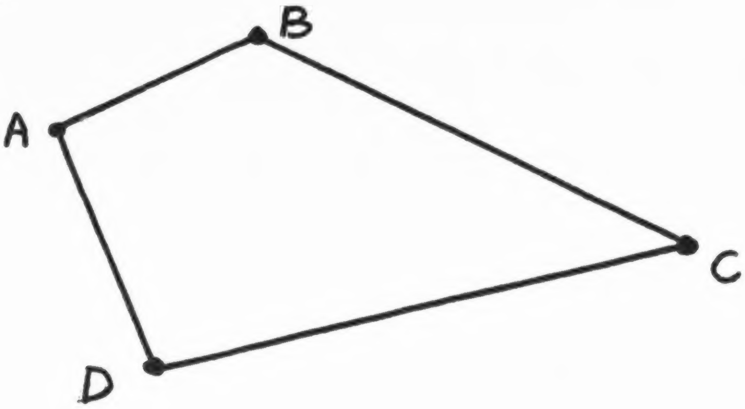


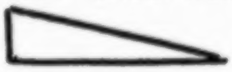
The union of three line segments is called a triangle. The union of four line segments is called a quadrilateral. Examples are:



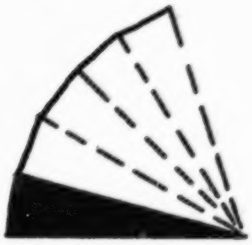
This is not a quadrilateral.

The end points of the segments of the figures above are called vertices. For example, the quadrilateral shown below has four vertices, one each at A,B,C, and D.

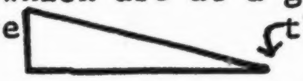


Next, consider a polygon that has many, many sides. One of the best ways to create this kind of polygon is to use a triangle shaped like this: 

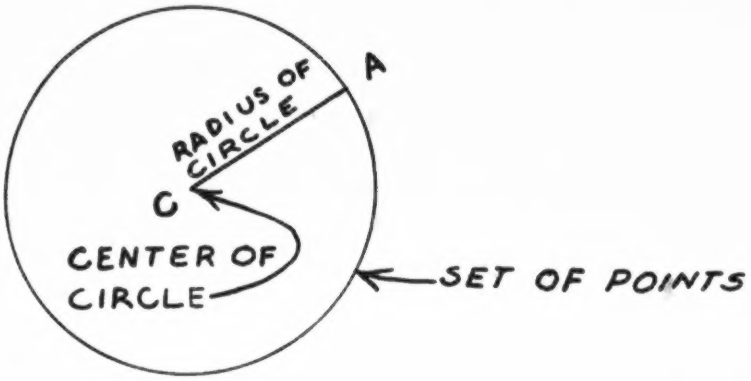
Think of the triangle being used over and over again as is depicted below:



If we were to select triangles with successively smaller bases,

we could conclude that we would finally produce a polygon with so many sides that it would be impossible to count them. Finally, we would arrive at a time when we would have nothing but a set of points all of which are at a given distance from another point. This point is where  this vertex of the triangle was placed in completing the figure below.

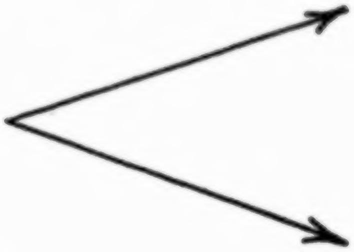
The name for this set of points is the circle.



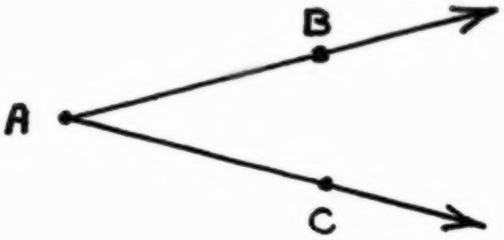
The set of points making up the radius and the center point are not a part of the circle. Only the set of points at the given distance CA from the center is a part of this model of a circle.

ANGLES CONGRUENCY

The familiar geometric representation shown below is called an angle:



The next thing necessary is to have a symbol with some method of naming the angle. Let's add three points A, B, and C.

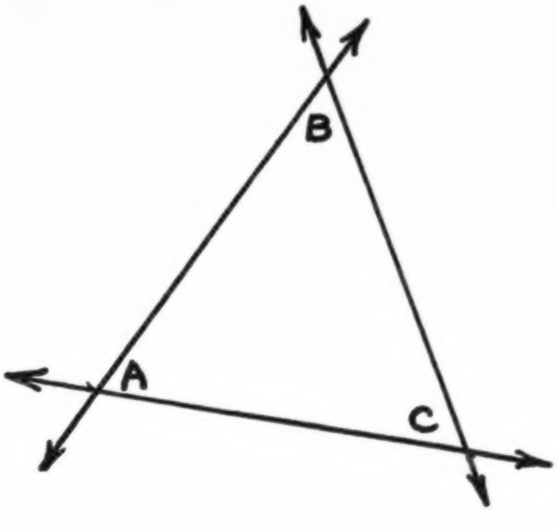


The symbol will be ∠ and we will use the points to name it. We write ∠ BAC (read "angle BAC"). It could also be labeled as ∠ CAB. (Note the position which the letter A occupies in both cases.)

A definition of the angle would be the union of two intersecting rays. The point where they intersect would be called the "vertex" of the angle.

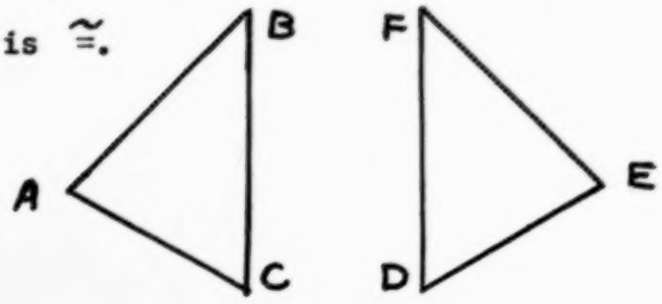
According to our definition of an angle we must think of the sides (line segments) of a triangle extended in order to have rays. Consider the following diagram:

The triangle has three angles (not a part of the triangle but just angles of the triangle). They are ∠ CAB, ∠ ABC, and ∠ BCA.



In our previous discussion of line segments, it was assumed that if two segments have the same exact measure (length) they are called congruent. The exactness of these measures or lengths may be determined, as we described earlier, by designing models of segments and placing them so that one coincides exactly with the other. This was our method of determining congruency. The method would also be the same for angles, triangles, quadrilaterals, etc.

The symbol for congruency is $\cong$.



Are the above two models of triangles congruent? How could we prove or disprove our supposition? If they are congruent we may write the following symbolic sentence: $\triangle$ ABC $\cong \triangle$ DEF. This would be read: "Triangle ABC is congruent to triangle DEF."

Students will have probably concluded by this time that geometric figures are congruent if every part of one exactly corresponds to that same part of the second. To this point we have been able to show this by superimposing one on the other to observe the degree of "sameness" or matching.

There are better, or at least other, ways of making such comparisons. One is called measurement.

MEASUREMENT

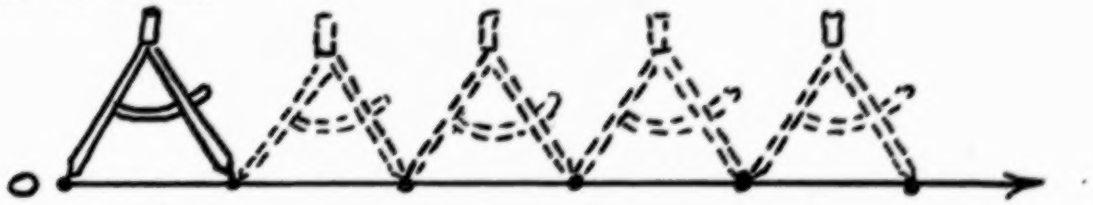
Measure compares the thing to be measured with a standardized and arbitrary unit. We will be concerned here with measurement in one dimension (linear), in two dimensions (angular and area), and three dimensions (volume).

The first problem in measurement is to establish a unit which can then be applied to the unknown quantity and which will provide an answer to the question "how much."

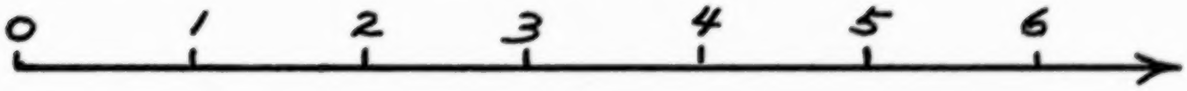
For linear measure the compass and a ray are necessary to create the measuring device most commonly used.

Let's assume we start with the ray shown as follows:

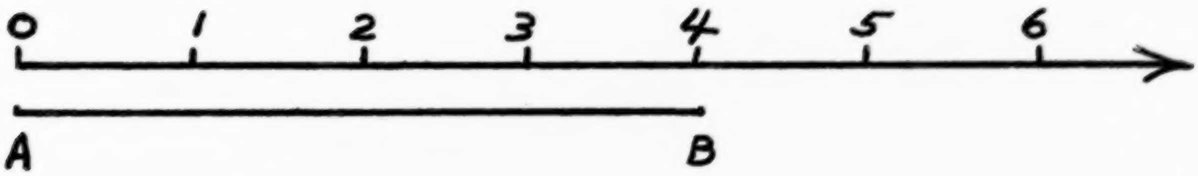
Next we have a compass opened to a particular setting. Beginning at the point 0, we mark off equal segments on the ray with the compass as shown.



Next we lable these points, starting with 0.



We are now ready to use our measuring device to determine an answer to how many units (or parts of units) of the size marked off on the ray is the measure of any unknown segment. This would be determined by bringing the segment alongside the labeled ray and answering the question.

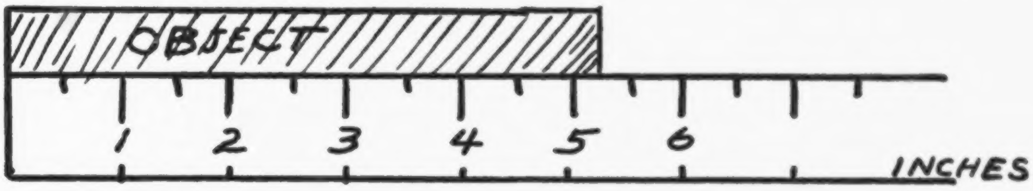


$\overline{AB}$ is 4 units long.

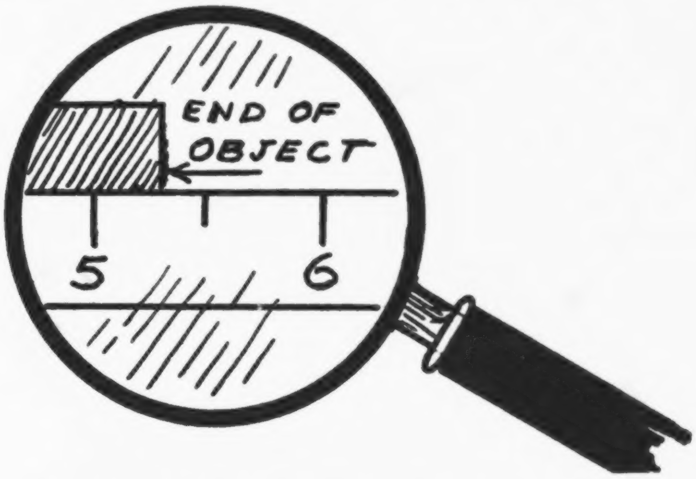
Of course, if we open the compass to mark off the ray in segments the length of 1 inch we have a device that will read linear measure in inches of parts thereof. If the compass were opened to mark off segments of 12 inches we have an instrument which will measure feet or parts thereof, and for yards, rods, etc., the same would be true.

It is desirable that every youngster in the primary grades should be acquainted with, and have available, a ruler to use in making linear measurements. A compass should also be available for use in the primary grades. (If care is exercised in using the common compass accidents from the points can be avoided.)

As a variety of segments or objects are measured it will become obvious to all that measurement with a ruler is approximate. For example:

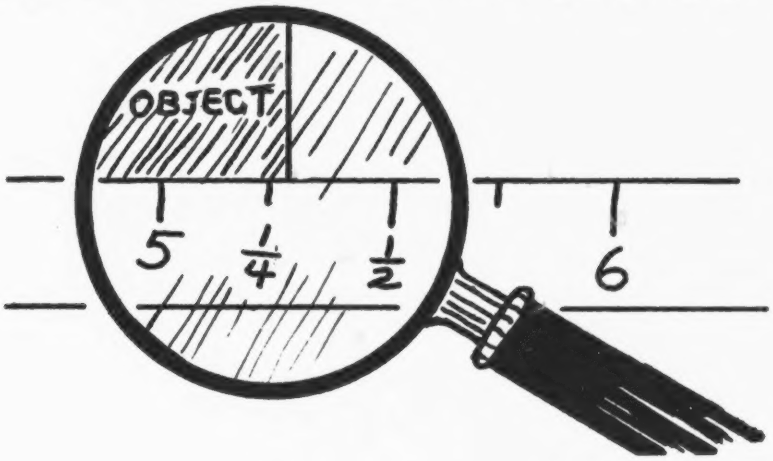


How long is the object? The object is clearly 5 inches long with some left over. Now let's look closely at that part left over.



Is the object nearer 5½ or 5 inches? Of course it is nearer 5½. Since we are using a ruler whose smallest segment is ½ inch, we are to report the length of the object to the nearest ½ inch and will say it is approximately 5½ inches long.

Let's assume that we have a ruler marked off in ¼ inch segments and look again at the object through the magnifying glass.



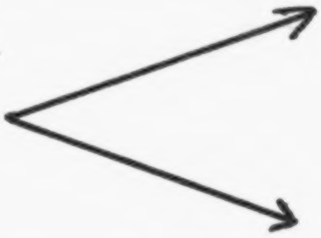
Now, is the object nearer $5\frac{1}{4}$ inches or $5\frac{2}{4}$ inches ($5\frac{1}{2}$ inches)? The answer is $5\frac{1}{4}$.

This procedure should be repeated, using rulers marked off in segments of $\frac{1}{8}$ inches and $\frac{1}{16}$ inches. Remember, each time the length should be reported to the nearest one of the smallest units into which the ruler is marked. If the ruler is marked off in half-inches, then the approximate measure can be reported to the nearest half-inch, quarter-inch, and so on.

From a series of laboratory exercises such as this the student will be led to generalize that the smaller the unit of measurement marked off on his ruler the more precise is the measurement and also that the greatest possible error (assuming the ruler is read correctly) is one-half the unit of measurement.

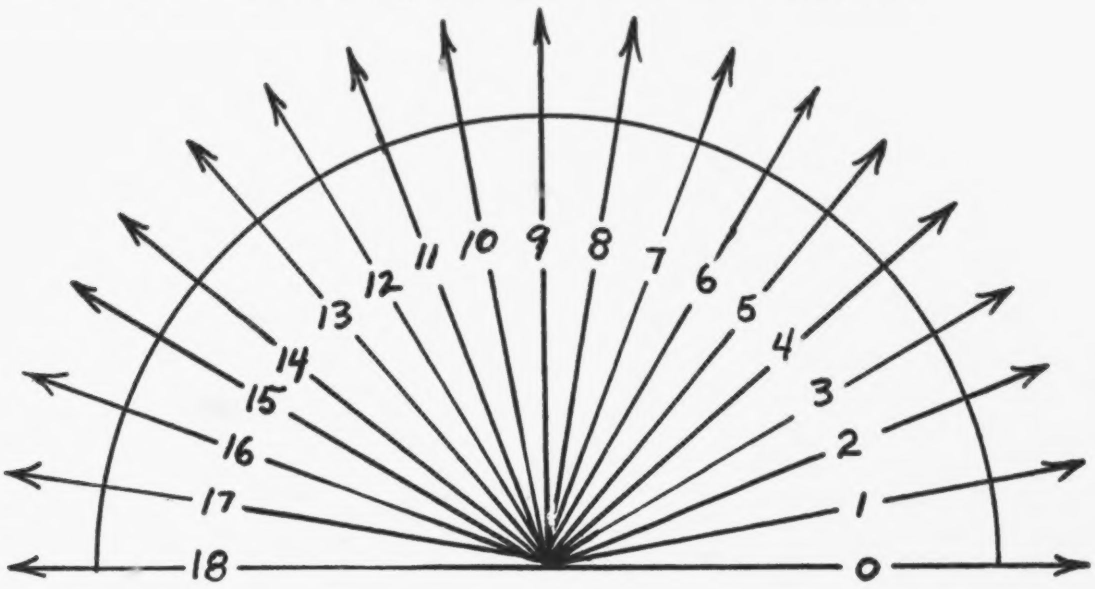
ANGLE MEASURE

Assume that we have the following model of an angle:

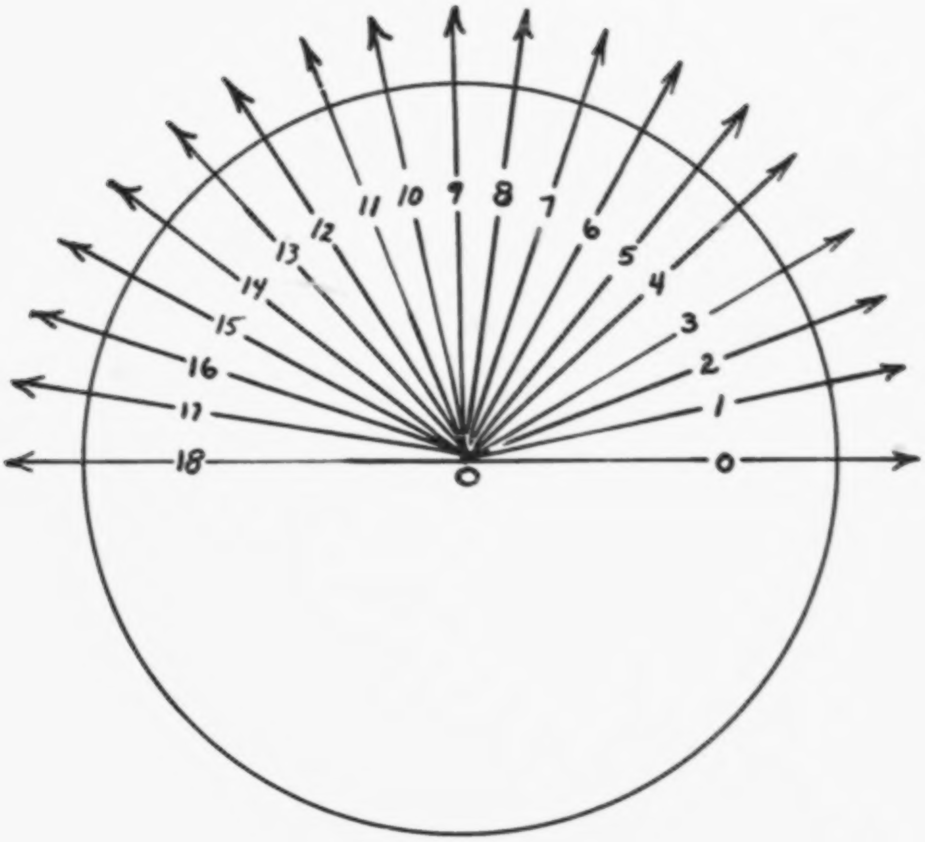


How could we measure the size of the angle above? The logical answer is by measuring in some manner the opening between the two rays. Then we must design an instrument which will give us a measurement of this opening formed by the two rays, which we call the angle. Remember, as in linear measure, we must first find some unit measure to apply to the unknown and then find the approximate number of these unit measures which is contained in the unknown.

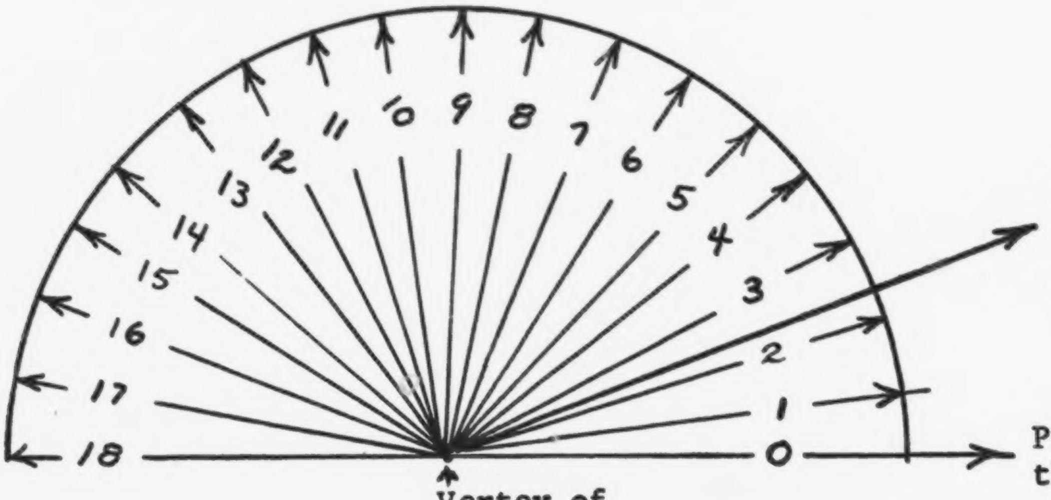
Consider the following collection of unit angles:



Next take the compass and using 0 as a center draw a model of a circle.


0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
0

If we cut this circle out we could have a half-circle marked in unit measures. The model of the given angle could also be cut out and placed on the measuring device as follows:


0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18

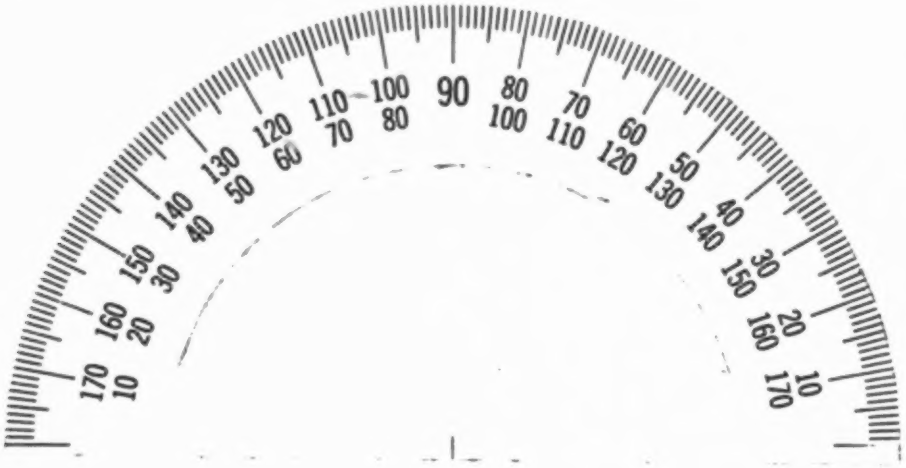
Other ray falls very near 2.

Place one ray of angle so that it exactly coincides with the ray marked 0.

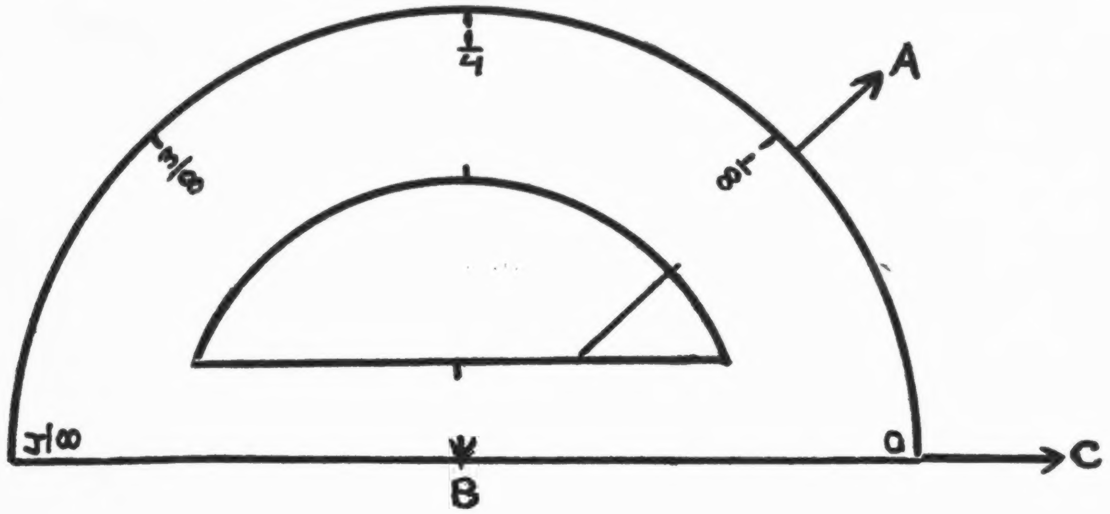
Vertex of angle at 0

We would report the measure of the unknown angle as approximately 2 units. If each of the 18 basic unit measures in the instrument we have just made is subdivided into ten equal units, we would have 180 instead of 18 units in the half circle. This, of course, is the common protractor which is marked off in 180 unit measures. Each measure is known as a degree and is written as a small circle to the right and above the numeral, as 20°. This would be read "twenty degrees."

The protractor looks like this and is divided into units known as degrees.

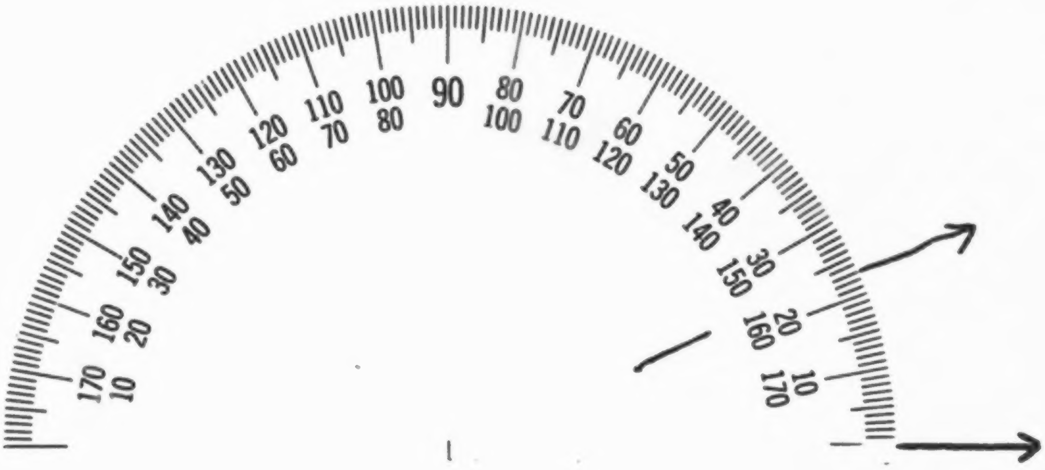

10
20
30
40
50
60
70
80
90
100
110
120
130
140
150
160
170
170
160
150
140
130
120
110
100
80
70
60
50
40
30
20
10

In order to develop basic understandings students may construct simple protractors by dividing a circle into halves, then quarters, etc. Compare a given angle in a manner which indicates whether it is "more than a quarter cirle," "about an eighth of a circle," and so forth. The instrument is used as illustrated below:


1/4
3/8
1/8
A
1/2
0
B
C

(1) Take a given model of an angle which is to be measured.
(2) Place the point on the straight edge of the protractor exactly over the vertex of the model angle.
(3) Be sure that one ray of the angle exactly matches the 0 of the scale on the protractor.
(4) To find the measure of the angle read the number of degrees on the scale exactly opposite the other ray (side) of the angle. (In this case, approximately 23°.) Again, the measure of the angle can only be reported to the nearest unit of measurement into which the protractor is subdivided.

Most protractors have two sets of numerals on the scale depicting degrees. This is added for convenience so that the approximate measure of the model angles can be read from either scale of the instrument. Both numerical scales have one point in common. Find that point on the following drawing.



As we see in the diagram, it is 90° (ninety degrees). This point measures angles of a very special size. Angles of 90° measure are called right angles. "Square" corners on buildings, books, paper, etc., all have a measure of 90°. A triangle which has an angle whose measure is 90° is known as a right triangle. What is the sum of the measures of three angles of any triangle? We can measure them with a protractor and find out.

Students in the elementary grades should have many laboratory experiences dealing with the measurement of angles with the protractor. The instrument is very inexpensive and can be acquired from many sources. Ambitious students may find it possible to construct their own protractor. Demonstration protractors for use on the chalkboard are also available and field protractors to use out of doors are useful in helping students to understand the measure of angles. The ultimate in a field protractor is a surveyor's transit, but a less complicated device for use in the elementary school is probably more practical.

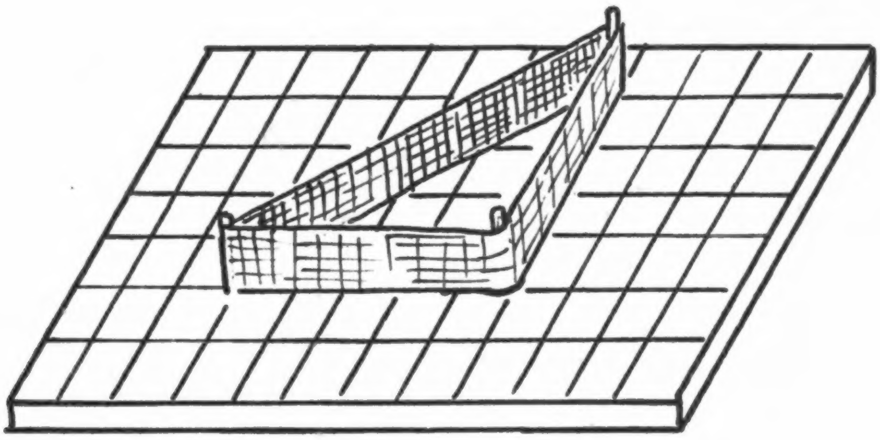
Teacher demonstrations of techniques for manipulating learning aids in the area of geometry are often unsuccessful because the devices are too small to be visible by students seated in the classroom. Various overhead projectors are valuable in overcoming this difficulty. Other aids such as place value charts, perimeter-area boards, and grids may be reproduced on a transparency and used by the teacher for demonstrations. Students can then be supplied with the devices for use at their seats. The overhead projector can be used by the teacher to direct the manipulation of the learning aid by the students. Ordinary ½-inch squared paper will serve as a grid for student use in developing perimeter and area concepts, and making arrays and number lines. Transparent geometric models and shapes, fractional rulers and other fractional representations, protractors, etc., are valuable for use with the overhead projector. It should be remembered that the use of the projector should result not in a viewing activity only on the part of the students, but in one in which they participate.

PERIMETERS

We may now recall our definition of a simple closed curve as being the path generated by going from a point such as A to another point B and returning by a different route. A triangle is an example of a model of a simple closed curve. The measure of a triangle or any simple closed curve would be the union of the line segments composing the curve. Another, perhaps more simple, way of saying this is: The measure of a simple closed curve may be found by adding the measures of each individual line segment. When we find the measure of a simple closed polygon we call this measure the perimeter. This measure of a circle is called the circumference.

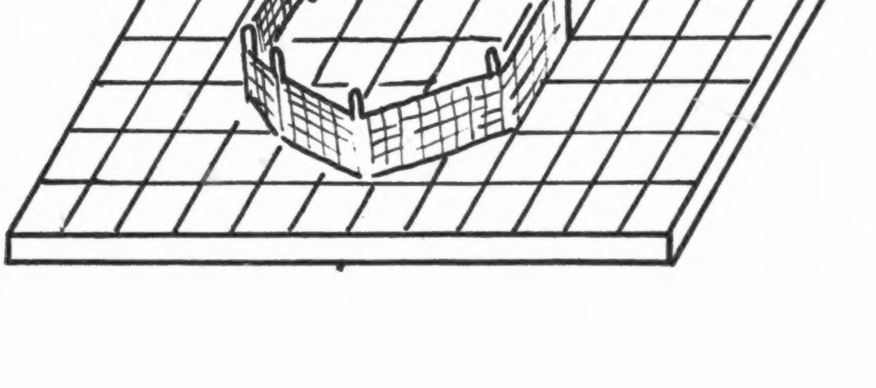
Before students are given formulas for finding the measures of perimeters and circumferences, they should have the opportunity to have many laboratory experiences dealing with these concepts. Many students will be able to discover formulas themselves, others will need assistance, but all should have the opportunity to try some of the experiments.

In working with the concept of the measure of perimeter many teachers have found a perimeter type corkboard to be very helpful. The perimeter board is a device which permits the drawing of a closed curve and which also is made of cork or a similar substance to receive pins at the vertices of the curve. The following diagram illustrates this:



After the tape measure is stretched around the pins it may be removed and the measure of the triangle approximated by reading the tape. It will also be discovered that the same quantity may be obtained by finding the linear measure of each of the three sides of the triangle and then adding these quantities. Consideration should be given to the degree of the precision of the measure as presented in the foregoing section or measurement. Polygons of any type may be experimented with in this manner.

Another interesting experiment is to create a closed curve which is as nearly circular as possible and use many pins to approximate a "many, many"-sided closed curve.



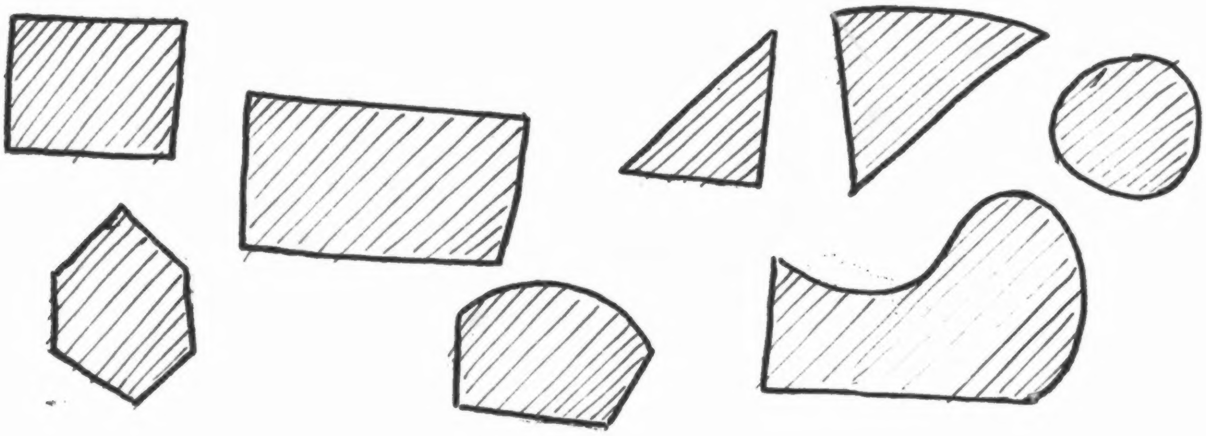
Stretch the tape measure around these pins, remove it, and read an approximate measure of the circumference of the circle. Next take the linear measure of the diameter (2 times the measure of the radius) of the circle and divide the measure of the circumference by the measure of the diameter. Repeat this for different-sized model circles. What is discoverable about all the quotients so obtained?

A similar experiment could be conducted by creating the models of circles out of wire and then straightening out the wire to make the measures. Teachers find a perimeter board a most valuable device in studying the measure of perimeter of many types of closed curves.

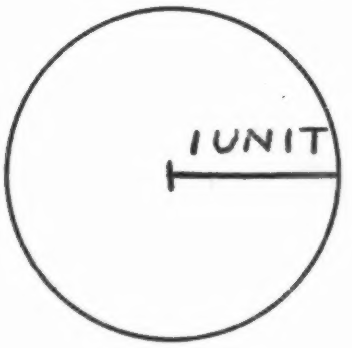
AREA

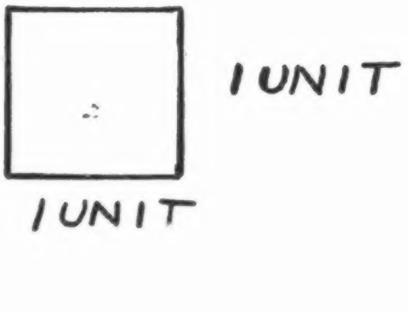
We have developed the ideas of finding the length of a segment by comparing it with a unit segment, and of finding the measure of an angle by comparing it with a unit angle. Now we will turn our attention to the matter of finding a unit measure for the internal region of a closed curve.

Consider the set of closed curves shown below. The problem with which we are concerned is to develop a unit measure which will, when applied to the shaded regions inside the closed curves, provide an answer to the question, "How much flat surface is contained inside this simple closed curve?" We will call the measure of this flat surface area.



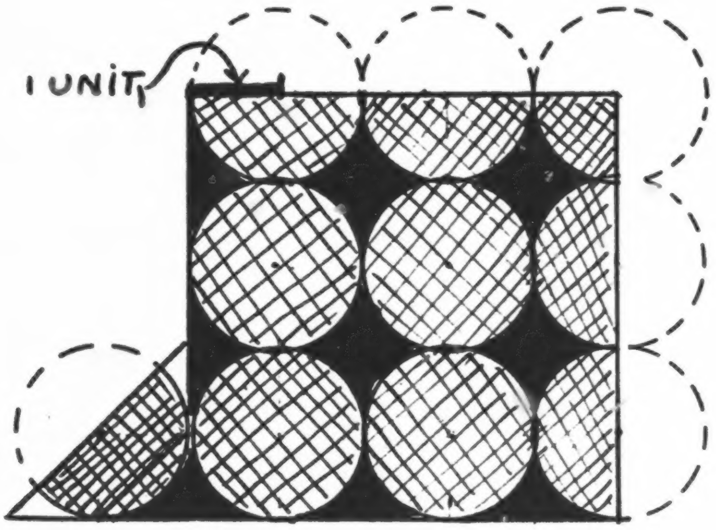
We would like to have whatever unit measure of area we create be easy to apply to as many of the closed curves drawn above as possible. Consider a unit measure as a circle with radius 1 linear unit and also a unit measure as a square whose measure is 1 linear unit on each side.





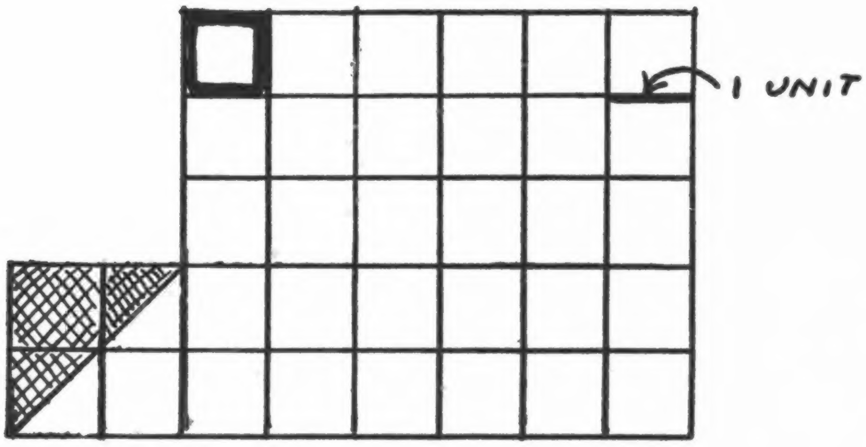
The question remains--which of the two unit area measures would be the easier to apply?

Consider the following situation: Suppose we have a closed curve such as that which is pictured and we will try to apply both unit area measures which we have designed.



The unit circle has been applied here.

The unit square has been applied here.



When the unit circle is used as a measure, note the area (blackened) which we still have not measured and also the fractional parts of unit circles (crosshatched) which we must consider.

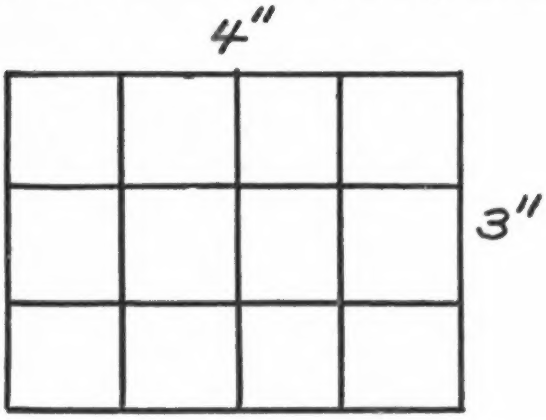
In using the unit square we have no area which has not been measured and we have only three fractional parts of unit squares (cross-hatched) which we must consider.

Now answer the question: In which part of the situation could the area be more accurately reported? There seems little doubt that the case in which the unit square was used furnishes the best report of the measure of the area contained inside the closed curve.

This kind of experiment should be tried by students using many different kinds of closed curves. It not only gives logic to area determination but in addition it provides opportunities for valuable discoveries relative to the basic meaning of area.

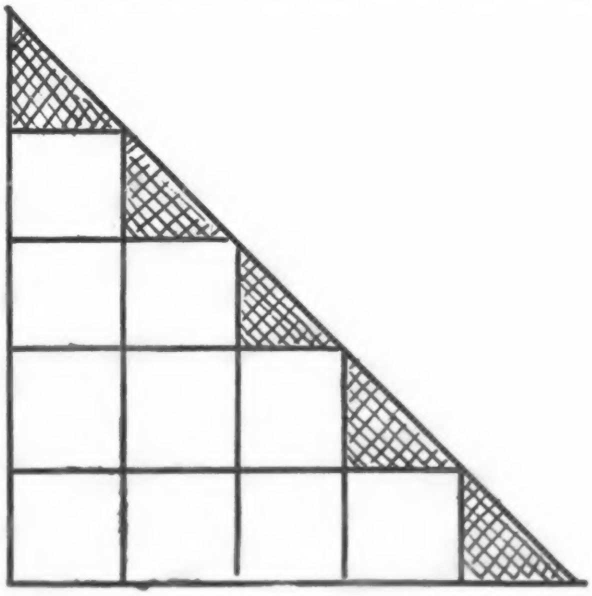
Since the decision has been made to use the unit square as the basic unit of area measure, we of course would report the area of a region within a simple closed curve as being so many square units. If the measure of the line segments of the unit square were 1 inch, the unit area would be "1 square inch"; if it were 1 centimeter, the unit area would be " 1 square centimeter"; if it were 1 foot, the unit area would be "1 square foot"; and so forth.

Assume that a model of a rectangle has a measure of 4 inches in length and 3 inches in width, what would be the measure of its area? How would this measure of area be reported?



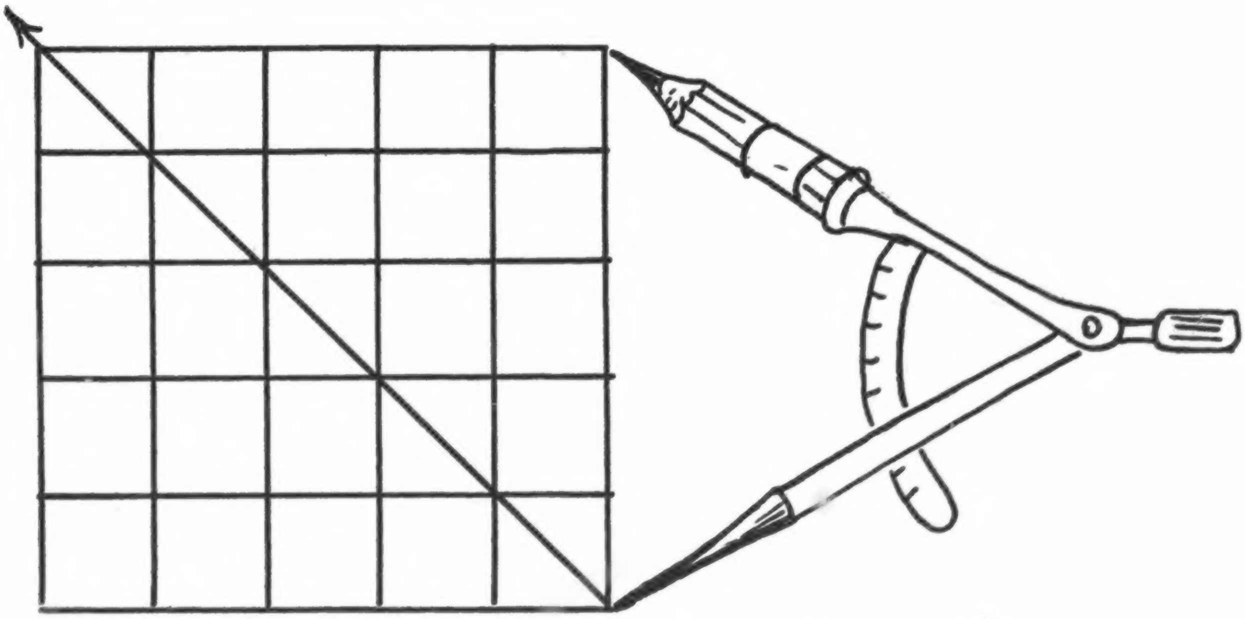
Of course, the logical unit of measure would be one square inch. We find by counting that the measure of the total area is 12 unit squares which, in this case, is 12 square inches.

Next, look at a model triangle such as that represented below.



The area of this triangle would be 10 square units plus some other fractional parts of square units. We could estimate these fractional parts as being about 5 halves of unit areas or an additional 2½ square units. Then the total area of the triangle would be measured as 10 + 2½, approximately 12½ square units.

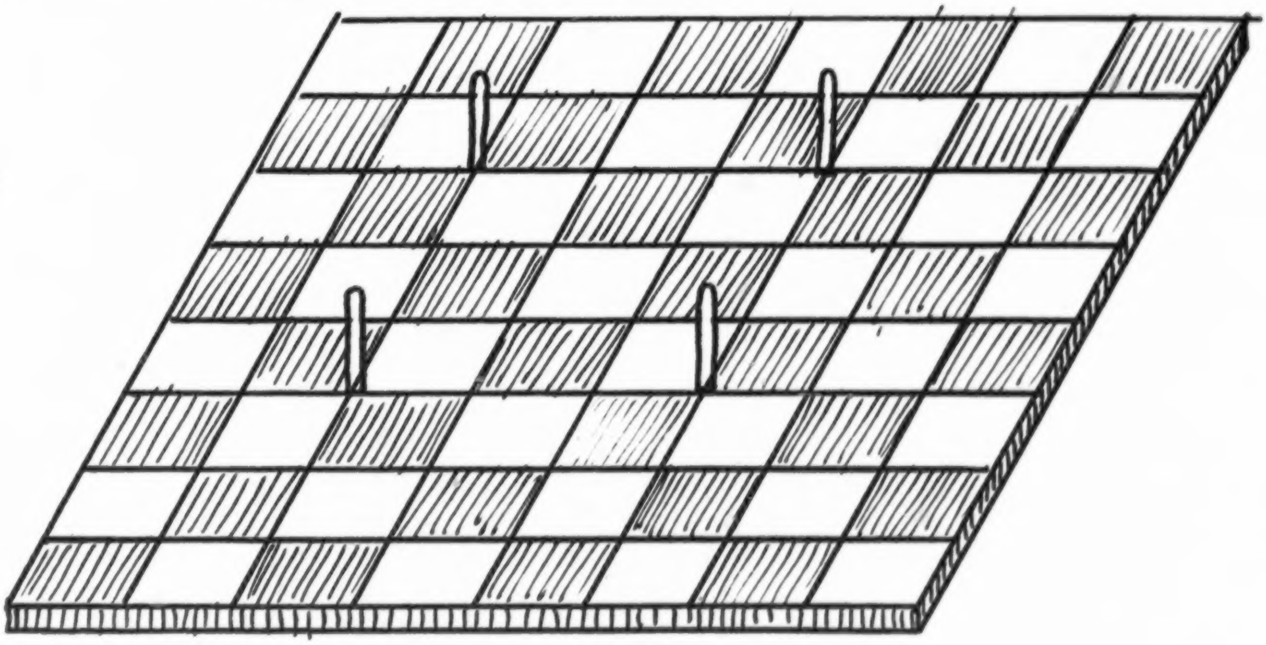
To carry this experiment one step further--use the compass to measure the segments sides of the triangle represented above and construct a quadrilateral with segments of the same measure. Of course this turns out to be a special kind of quadrilateral. What kind?



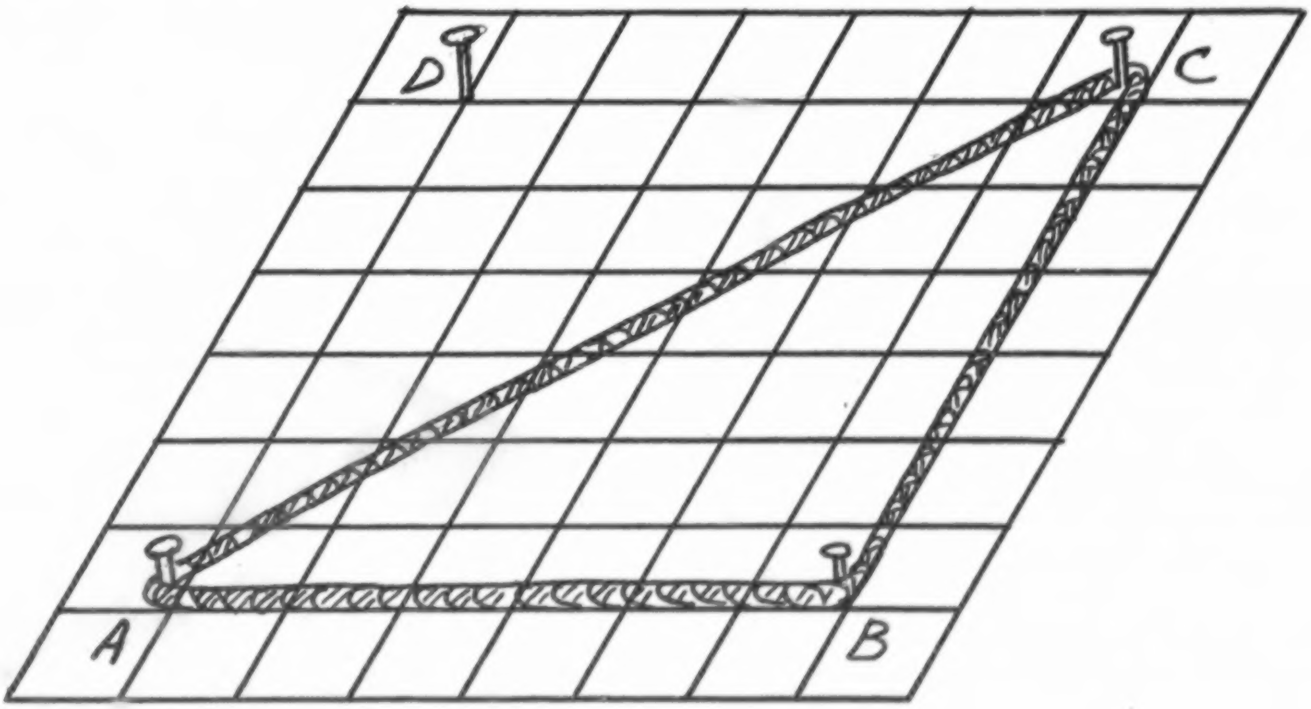
Now apply the square unit of measure to the area within the quadrilateral. The measure of the total area of the quadrilateral is 25 square units. How is this related to the area of the triangle used in constructing the quadrilateral? (Most students at grade levels 5 or 6 will be able to discover the relationship of the measure of the area of a triangle to that of the quadrilateral constructed from the triangle.)

Exercises of this type should be used with students until they are able to discover the fact that the measure of area can be determined by finding the linear measure of the segments composing the sides of the geometric figure and then treating this in a special way. For the square, this would simply be finding the linear measure of a segment forming the sides of the square, then squaring this number; for a rectangle, finding the product of the numbers obtained by measuring the lengths of two adjacent sides of the rectangle; for a triangle, finding ½ the product of the lengths of the adjacent sides of the superimposed rectangle.

To assist students in this discovery process, the "perimeter-board," described in the previous section (with the addition of square units outlined on the board), is a very valuable device. It can be used for demonstrating the measure of perimeter and area and for constructing arrays as described in the section on operations. Aids of this type can also be adapted for use as number lines.



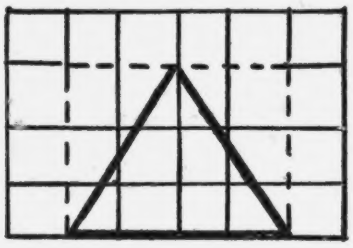
The board would be used as follows for experimentation with the area of the triangle:

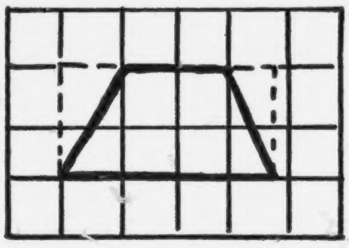


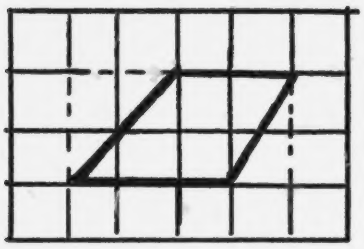
(1) Locate a pin at A.
(2) Place another pin at B.
(3) Place a third pin at C.
(4) Stretch a rubber band or elastic thread around A, B, and C.
(5) Approximate the area by counting the number of squares of unit area in the region inside the rubber band.
(6) Next, locate another pin at D.
(7) Stretch another rubber band around ABCD.
(8) How is this area related to the area of the triangle?

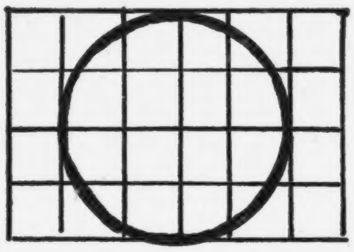
The same procedure can be followed with models of parallelograms, trapezoids, circles, etc.

The following are examples of these experiments as they would appear if the perimeter area board were used to develop the concept of area.









There are many applications of the area concept in everyday life. However, rather than relying on "floor covering" and "painting," we find classroom experimentation with geometric models, and student activities such as a survey of the school yard, with related problem situations, more satisfactory.

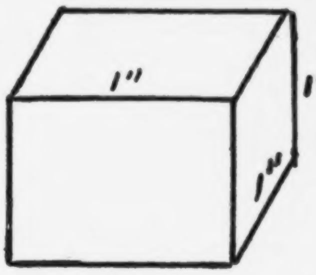
VOLUME

Up to this point we have covered measurement in one dimension (linear) and measurement in two dimensions (angles and area), and we need to look briefly at measurement in three dimensions (volume).

3reg.

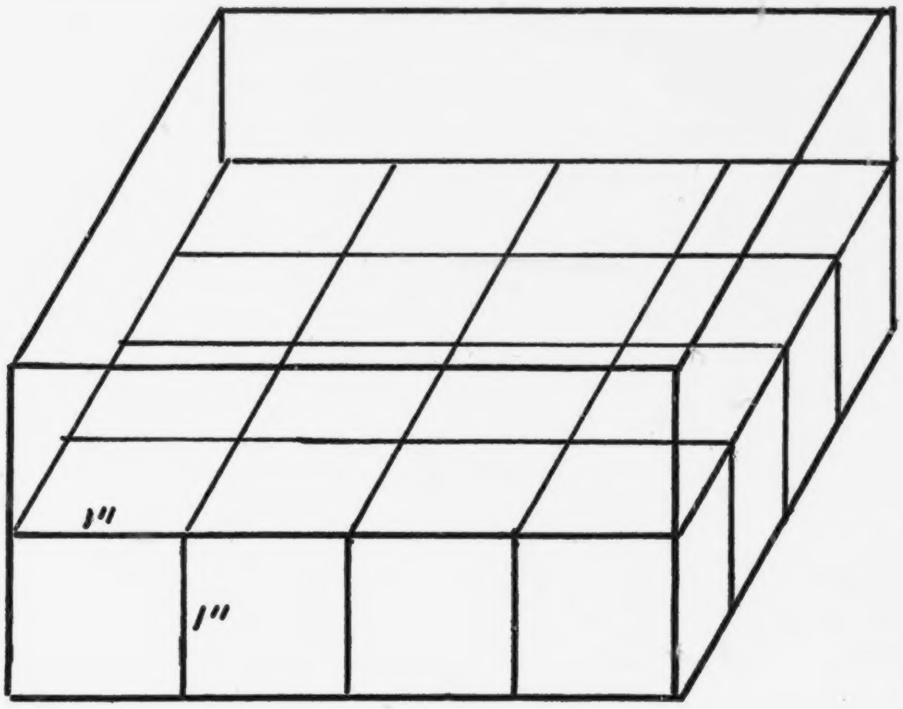
The first essential is to develop a measure of unit volume, since the concept of volume involves stating the amount of space contained within (the interior of) a box, a cone, a sphere, a pyramid, and so on.

A unit measure for volume will be a cube whose linear measures are 1 unit long, 1 unit wide, and 1 unit high.



This will be referred to as the measure, 1 cubic unit. If the linear measure of a solid is in inches, then the cubic measure can be expressed in cubic inches.

Now that we have a unit of measure for volume and the concept of what we mean by the word volume, we will proceed to apply this knowledge to determine the volume of a rectangular solid (box). We will need to use a solid with dimensions in even inches so that the unit measure (cubic inch) will fit. These are often difficult to secure but can be constructed. Consider a box whose dimensions are 4 inches long, 4 inches wide, and 2 inches high. We note that 16 of the cubic-inch blocks are required to cover the bottom of the box.

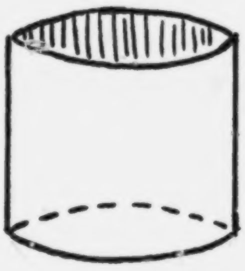


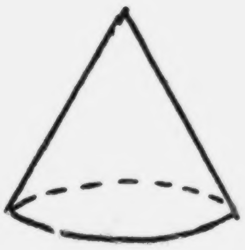
Since the box is 2 inches high, the first layer comes up to 1 inch. A second layer can be placed in the box on top of the first layer. Then the box contains 2 times 16 blocks or 32 cubic inches. The measure of the solid 4 inches by 2 inches by 4 inches is 32 cubic inches.

Experiments such as this are helpful to students in understanding the concept of volume. Students should be encouraged to discover the method of computing volume of various solids and then to develop formulas to guide their computation, rather than to memorize formulas and then to apply them without a real understanding.

A set of models of three-dimensional solids for individual student experimentation is essential to teaching the concept of volume. Such solids should be constructed so that measures of the unit volume can be stacked inside for the purpose of approximating the measure of the volume. They should also be made in such a way that they will hold liquids for the determination of volume.

For example, consider a cylinder and cone having equal height and area. Fill the cone with liquid and empty it into the cylinder.

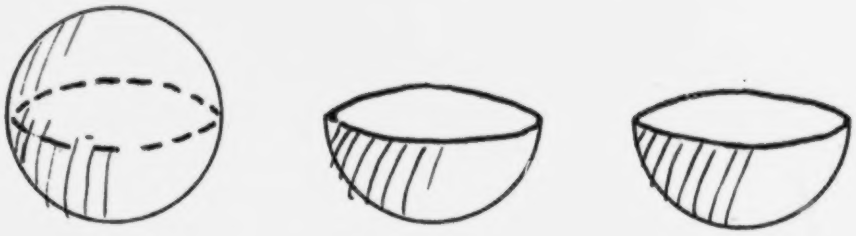




Students should experiment to find the number of cones required to fill the cylinder, then to develop a formula for the determination of the volume.

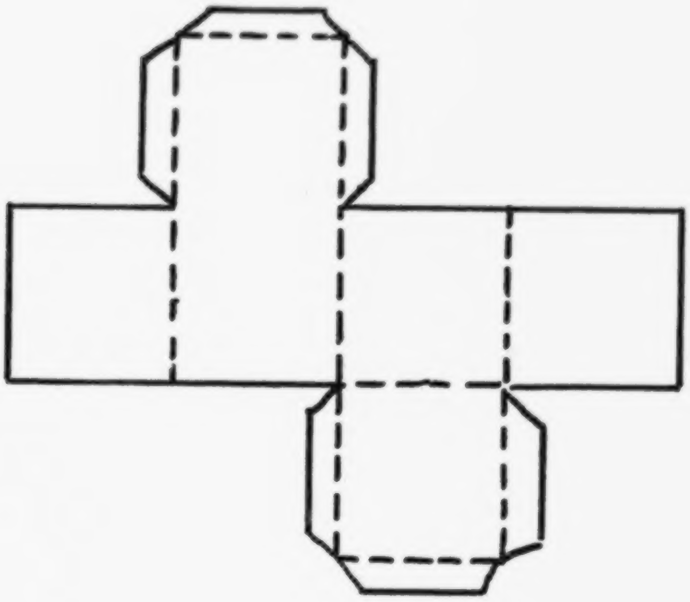
In order to approximate the measure of the volume of a solid such as a cylinder, other units must be used rather than small blocks. Since the inside surface of the cylinder is curved, the rectangular solid blocks will not fit exactly. This experiment has to be conducted with something that will take the shape of the inside of the cylinder. A liquid such as water will do this. We should conduct the experiment by filling one cubic unit measure with water and emptying it into the cylinder, counting the number of cubic unit measures required to fill the cylinder. Students should then determine the relationship of the measure of the volume of a cylinder to the measure of the circular area contained in the base of the cylinder.

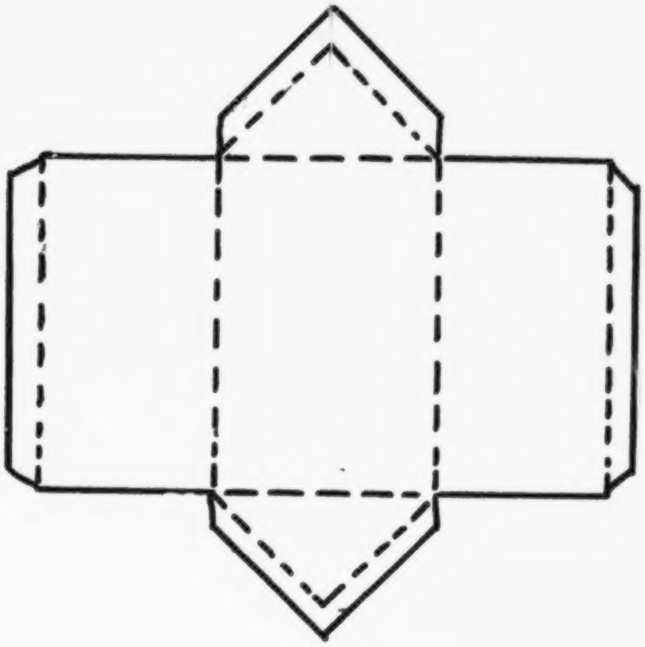
The volume for a sphere may be approximated in the following way:

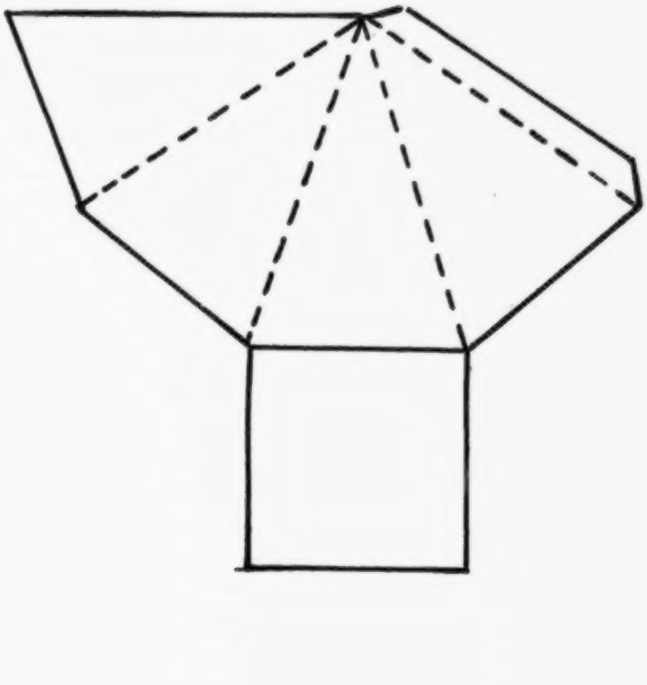


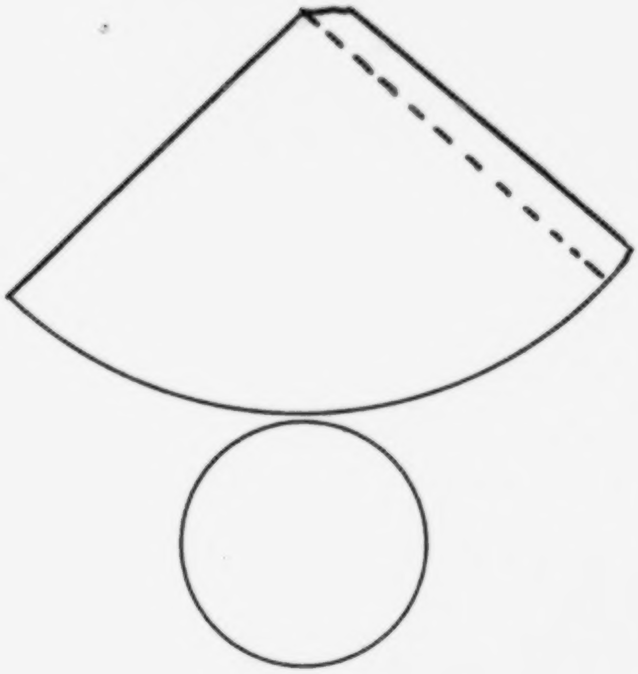
Take the sphere apart and fill each half with liquid as in the case of the cylinder. Discover the relationship between the measure of the volume of a sphere and the measure of the area of the largest circle which could be drawn on the sphere.

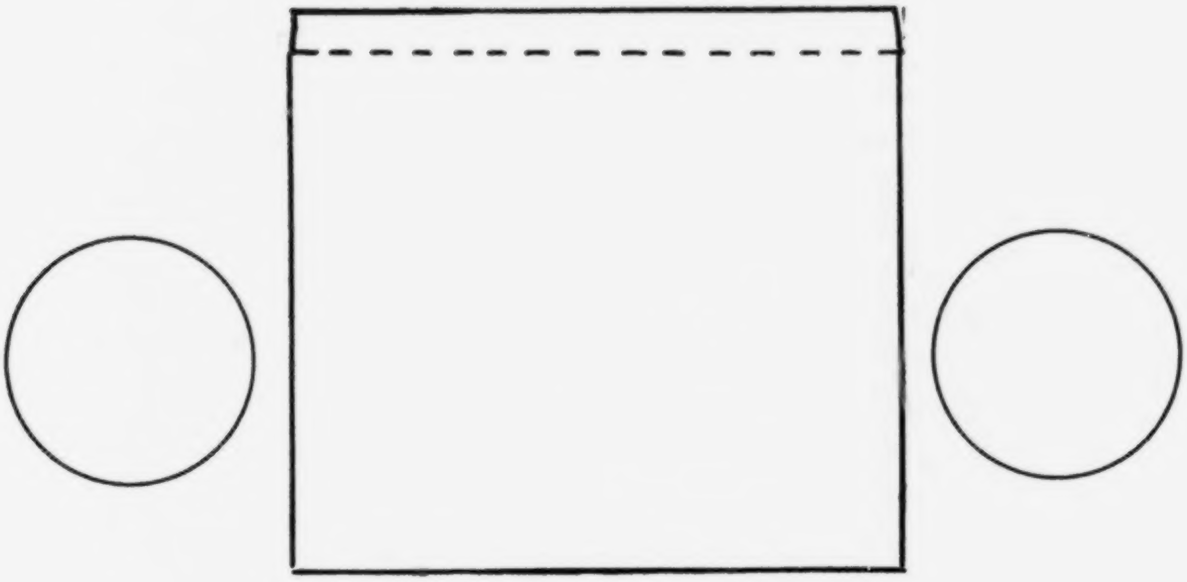
Another kind of experimentation which is quite appropriate for this level is drawing the surface of a solid on a plain sheet of paper in such a way that the paper can be cut, folded, and pasted together to form a three-dimensional model. Many patterns for this type of construction are available. A few examples follow, which may be traced and constructed. Making patterns for construction can be an excellent beginning experience with drawing instruments. Capable students should be encouraged to design their own patterns.











The kind of experimentation provided by cutting out patterns and pasting them together is not worth the time and effort involved unless special attention is given to the development of the mathematical ideas which are a part of this exercise.

Appropriate questions to ask along with such experimentation are:

(1) How many faces has this model?
(2) Why must the linear dimensions of certain faces be made to match?
(3) What is the measure of the area of various parts?
(4) What is the measure of the total surface area of the model?
(5) What is the relationship between the measure of the volume and the measure of the surface area of various faces?
(6) If the model were sliced with a sharp knife in various ways, what would the cross section look like?

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